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# PROPORTIONALITY IN TWO DIMENSIONS

Applied to the representation of Cantons and Constituent Peoples in the
House of Peoples of the Parliament of the Federation of Bosnia and Herzegovina

## 1 The formal framework

There are given $H$ units of some good, where $H$ is a positive real number. The units are to be distributed among a number of entities.[1]

The entities are characterized by two attributes. Therefore, they can be arranged in a two-dimensional matrix. Let $I$ and $J$ be the number of rows and columns in the matrix. Hence there are a total of $I \cdot J$ entities competing for the $H$ units of the good. The variables $i$ and $j$ vary over the sets $M = \{1, 2, ... I\}$ and $N = \{1, 2, ... J\}$, respectively.

There are given positive real numbers $p_{i,j}$. They represent the entities' claims on a share of the $H$ units of the good. As far as possible the distribution should be proportional to the numbers $p_{i,j}$.

Row and column sums in the $p_{i,j}$-matrix, and the sum of all entries in the matrix, are denoted as follows:

$$p_{i,N} = \Sigma_{j \in N}\, p_{i,j}$$
$$p_{M,j} = \Sigma_{i \in M}\, p_{i,j}$$
$$p_{M,N} = \Sigma_{i \in M, j \in N}\, p_{i,j}$$

Similar notation is used for other indexed variables.

It is convenient to *normalize* the numbers $p_{i,j}$ by defining

$$q_{i,j} = p_{i,j} \cdot H / p_{M,N} \text{ for all } i \text{ and } j$$

Then $q_{M,N} = H$.

Let $a_{i,j}$ denote the amount of the good allocated to entity $(i, j)$. These must be non-negative numbers satisfying $a_{M,N} = H$. The $a_{i,j}$-matrix is called an *allocation*. For now, there is no requirement that $a_{i,j}$ be an integer.

Perfect proportionality can be achieved by letting $a_{i,j} = q_{i,j}$. If there are no further constraints, this is the obvious solution to the problem.

Further restrictions are imposed. There are given positive numbers $r_i$ for $i \in M$ and $s_j$

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1. The word "Entity" (usually with a capital initial) has a special meaning in Bosnia and Herzegovina; see the first paragraph of Section 2. Here "entity" is used in a different and more general sense. Hopefully, this will not cause confusion.

for $j \in N$ satisfying $r_M = \Sigma_{i \in M}\, r_i = H$ and $s_N = \Sigma_{j \in N}\, s_j = H$. It is required that the allocation satisfy (1) and (2):

(1) $a_{i,N} = r_i$ for all $i$

(2) $a_{M,j} = s_j$ for all $j$

There will always exist allocations satisfying these constraints. One possibility is $a_{i,j} = r_i \cdot s_j / H$, but this could be far from proportional to $p_{i,j}$.

The task is to find an allocation, that is, a non-negative $a_{i,j}$-matrix, which as far as possible is proportional to $p_{i,j}$, given that (1) and (2) must be satisfied. Conditions (1) and (2) have absolute priority over the criterion of proportionality. That is, one is not permitted to make the slightest deviation from (1) or (2) even if this could significantly improve the proportionality between $a_{i,j}$ and $p_{i,j}$.

It may be impossible to find an allocation which is "fair" or "reasonable" in all relevant respects. Assume, for example, that an allocation is proposed in which, for some values of the variables $i$, $j$, and $j'$, $a_{i,j} < a_{i,j'}$ although $p_{i,j} > p_{i,j'}$. That is, within row $i$ of the matrix, entity $(i, j)$ has a stronger claim on the good than $(i, j')$, but nevertheless $(i, j)$ receives less than $(i, j')$. This seems unfair, and one might be tempted immediately to reject the proposed allocation, but such a reaction would be unjustified. It is possible that no allocation satisfying the constraints is free from this type of unfairness (or a similar type relating to entities in the same column).

In many interesting cases the units of the good are indivisible, so each entity must receive a whole number of units. The number of units given to entity $(i, j)$ is in this case denoted $h_{i,j}$. It plays the same role as $a_{i,j}$ did earlier, but $h_{i,j}$ must be a non-negative integer. This is referred to as the *discrete* case, as opposed to the *continuous* case previously discussed, where the good is assumed to be perfectly divisible. In the discrete case, the units of the good are usually called *seats*, since this corresponds to many important applications, included the one presented in Section 2 and discussed in later sections. Of course, the theoretical discussion is independent of the nature of the good to be distributed, except that it makes a difference whether it is divisible or not.

The $h_{i,j}$-matrix is called an *apportionment*. It must satisfy analogous versions of (1) and (2):

(3) $h_{i,N} = r_i$ for all $i$

(4) $h_{M,j} = s_j$ for all $j$

This is only possible when $r_i$ and $s_j$ are integers. Then $H$ must also be an integer.[2]

The discrete case is more restrictive than the continuous one. Therefore, it may all the more be impossible to find an apportionment which is "fair" or "reasonable" in all relevant respects.

The *data* of the problem are the numbers $I$, $J$, $H$, $p_{i,j}$, $r_i$, and $s_j$.

Clearly, $I$ and $J$ must be positive integers. If $I = 1$, the problem is not really two-dimensional, and the solution is determined by (2) or (4). Hence nothing is lost by assuming $I \geq 2$. Similarly, $J \geq 2$ can be assumed.

By assumption, $H$, $p_{i,j}$, $r_i$, and $s_j$ are positive numbers.

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2. When $r_i$, $s_j$, and $H$ are positive integers satisfying $r_M = s_N = H$, there will exist apportionments satisfying (3) and (4). The proof of this has limited interest and is omitted.

The *solution* is given by the numbers $a_{i,j}$ or $h_{i,j}$. They are required to be non-negative, but it is not *a priori* assumed that they are strictly positive. The solution to the continuous problem, given in Section 5, has $a_{i,j} > 0$ for all $i$ and $j$, but that is a conclusion, not an assumption. In the discrete case, the possibility $h_{i,j} = 0$ for some $i$ and $j$ cannot be ruled out. If $H < I \cdot J$, some $h_{i,j}$ must necessarily be zero.

The theoretical discussion in Sections 5 and 7 is based on two articles by M. L. Balinski and G. Demange:

[1] "An axiomatic approach to proportionality between matrices", *Mathematics of Operations Research* **14** (1989), 700-719.

[2] "Algorithms for proportional matrices in reals and integers", *Mathematical Programming* **45** (1989), 193-210.

## 2 The application

The Federation of Bosnia and Herzegovina is one of the two Entities that make up the internationally recognized state Bosnia and Herzegovina. The other Entity is the Republika Srpska.

The Parliament of the Federation consists of two houses. Only one of these, the House of Peoples, is of interest here. The goods to be distributed are the seats in the House of Peoples, of which there are a total of 80. Hence $H = 80$.

The Federation is geographically divided into *ten* Cantons.[3] The population of the Federation consists of *three* Constituent Peoples, Bosniacs, Croats, and Others. This defines the two-dimensional structure of the problem.

A Constituent People in a Canton is called a *group*. They correspond to the "entities" of the general discussion in Section 1. There are a total of 30 groups.

For indexed variable like $p_{i,j}$ etc., the first index refers to the Cantons, which are identified by their official numbers. The second index is used for Constituent People, with $j = 1$ for Bosniacs, $j = 2$ for Croats, and $j = 3$ for Others. Hence $I = 10$ and $J = 3$.

The Draft Election Law Article 12.3 second paragraph mandates that the 1991 Census be the basis for distributing the seats in the House of Peoples among the Constituent Peoples within each Canton. Therefore, the numbers $p_{i,j}$ are the population of the groups according to the 1991 Census.

Table 1 contains the population, according to the 1991 Census, of the Cantons, the Constituent Peoples, and the groups. Table 2 shows the "normalized population", that is, the numbers $q_{i,j}$.

The problem is obviously of the discrete type. A solution is an apportionment, that is, an $h_{i,j}$-matrix of non-negative integers, where $h_{i,j}$ is the number of seats given to Constituent People $j$ in Canton $i$.

The Draft Election Law contains, in Article 12.3 first paragraph, a procedure for distributing the seats in the House of Peoples among the Cantons. The idea is that the 80

3. The Brčko District is ignored. Presumably, the citizens of the Federation in the Brčko District should, for the present purpose, be included in one of the Cantons and be given an opportunity to influence the election to the House of Peoples from that Canton.

seats shall be distributed proportionally among the Cantons based on recent voter registration figures, but each Canton shall be guaranteed at least *three* seats. This determines the numbers $r_i$. The next-to-last column of Table 2 contains the numbers $r_i$ on which the computations in Sections 3 – 7 are based.[4]

Moreover, the Draft Election Law specifies, in Article 12.3 third paragraph, the number of seats in the House of Peoples for each Constituent People. This determines the numbers $s_j$. They are shown in the next-to-last row of Table 2.[5]

It is obvious from Table 2 that the row and column sums of the $q_{i,j}$-matrix are significantly different from $r_i$ and $s_j$. Therefore, even if the problem had been of the continuous type, considerable deviations from proportionality would have had to be accepted. In addition comes the restrictions caused by the discrete nature of the problem.

In part, the deviations are caused by the fact that different principles are applied when $r_i$, $s_j$, and $h_{i,j}$ are determined. The numbers $r_i$ are the result of proportional distribution, but based on different data than the ones that shall be used to determine $h_{i,j}$. The numbers $s_j$ are specified in the law. It is likely that better proportionality between $h_{i,j}$ and $p_{i,j}$ could have been achieved if the same principle had been used in all cases. The only practical possibility would be to determine $r_i$ and $s_j$ by proportional distribution based on the 1991 Census.[6] The last column and the last row of Table 2 contain the results of distributing seats among Canton and Constituent Peoples on this basis.[7] These numbers play no role in the subsequent computations.

## 3 A simple and straightforward solutions

The following appears to be a natural and straightforward solution to the discrete problem: Choose one of the well-known and commonly accepted methods of proportional representation. Distribute the seats among the entities (groups) according to this method, taking account of the constraints (3) and (4).

For some methods, like the method of the largest remainder, it is not clear what it means to "take account of" certain constraints when distributing the seats. Any attempt to make this

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4. In fact, the procedure of Article 12.3 first paragraph is inconsistent. The underlying idea can, for example, be realized by using a constrained version of the odd numbers method of proportional representation. This procedure gives the numbers $r_i$ of Table 2.

5. The law contains an additional restriction on the apportionment: A Constituent People that is represented by at least one member in a Cantonal Assembly, shall have at least one seat in the House of Peoples from that Canton. This rule is ignored in the computations below. The results of the Cantonal Assembly elections must be known before it can be applied.

6. More recent data, such as voter registration figures, do not contain information about Constituent Peoples and can therefore not play the role of $p_{i,j}$.

7. The numbers $s_j^*$ are the result of a straightforward proportional distribution. They are different from $q_{M,j}$ only because they must be integers. The $r_i^*$-vector is determined by a procedure of constrained proportional distribution, where each Canton is guaranteed at least three seats. Therefore, $r_i^*$ is not really proportional to $q_{\mathrm{i,N}}$, but the proportionality is better than that between $r_i$ and $q_{\mathrm{i,N}}$.

precise creates new difficulties or leads to paradoxical results, which must in turn be taken care of. It may be possible to find a satisfactory solution, but it must necessarily be quite complicated.[8]

The class of *divisor methods* is characterized by the seats being handed out one by one in a definite order. Then it is easy to take account of the constraints (3) and (4). If the awarding of a seat according to the divisor method will lead to a Canton getting more seats than permitted by (3), or a Constituent People getting more seats than permitted by (4), this seat is simply omitted, and one goes on to the next seat. An apportionment satisfying the constraints will eventually be found.

One specific divisor method, the odd numbers method, is used here.[9] It has the property of being neutral in relation to big and small entities, that is, on the average it does not favors the big ones compared to the small ones, nor vice versa.[10] It is also the divisor method which is closest to the method of the largest remainder. In particular, the two methods coincide when seats are distributed between only two entities.

To sum up, the following procedure is used: The point of departure is the numbers $p_{i,j}$, that is, the groups' population according to the 1991 Census. (Since normalized population is proportional to population, $q_{i,j}$ might as well have been used as the point of departure.) Seats are distributed on this basis by the odd numbers method. If the awarding of a seat to a group would lead to the corresponding Canton getting more seats than permitted by (3), or the corresponding Constituent People getting more seats than permitted by (4), this seat is ignored and one proceeds to the next seat in the order determined by the odd numbers method.

The distribution is documented in Tables 3 and 4. The groups are identified with a letter for the Constituent People and a number for the Canton. For example, group C 7 consists of the Croats (Constituent People 2) in Canton 7. In the notation previously introduced, this is entity (7, 2).

Table 3 consists of three parts, one for each Constituent People. For each group the table shows the population $p_{i,j}$ and the quotient obtained by dividing it by 1, 3, 5, and so on. It is indicated which seat, if any, is won on the basis of a quotient. The table contains the quotients that are used for distributing seats, and at least one more for each group.

Table 4 shows the awarding of the seats one by one. The columns "G", "P", and "C" contain the number of seats hitherto won by the group, the Constituent People, and the Canton, respectively.

When the 56th seat has been handed out, Constituent People 1 (Bosniacs) has gotten the number of seats to which it is entitled by condition (4). A couple of more seats can be awarded before the constraint really becomes binding. Up to and including the 58th seats the odd numbers method can be applied without modifications. However, quotients no. 59 cannot

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8. The reason is that the method is vulnerable to the so-called "Alabama paradox": If the total number of seats to be distributed is *increased*, everything else being kept constant, some entities can get *fewer* seats than before. The method is described and an example of the paradox is given in the Appendix.

9. The method is described in the Appendix.

10. In order to make this statement precise, the probability distribution over which the average is taken must be specified. The issue is not discussed further.

be used to award a seat. The same is true for several subsequent quotients. These quotients are included in Table 4, and in Table 3 they are marked with a number in brackets. For example, the seventh quotient for group B 4 is marked "(61)" to indicate that it lies between the quotients on the basis of which the 60th and the 61st seat were won.

The 79th quotient is used to hand out the 68th seat. Then Constituent People 3 (Others) has also gotten the number of seats to which it is entitled by condition (4). Therefore, the remaining *twelve* seats must all be given to Constituent People 2 (Croats). The distribution of these seats among the Cantons follows from condition (3).[11] Hence the final apportionment is determined, and further computing and comparing of quotients is pointless.

Table 5 shows the distribution of the 68 first seats, and Table 6 gives the final apportionment.

Although this procedure may seem natural, there are serious problems with it. The distribution of the last seats is determined solely by conditions (3) and (4), with no reference to population or proportionality. In the specific example, this is the case for as many as *twelve* seats.[12] Tables 5 and 6 show that group C 5 gets *two* seats, none of them won in the ordinary way on the basis of quotients. The group has a tiny population; see Table 1. If one looks at the Constituent Peoples within Canton 5, it is unsatisfactory that a distribution which pretends to be proportional, should give group C 5 two out of three seats. A similar argument can be made concerning the relationship between the groups C 1 – C 10 within Constituent People 2 (Croats). As pointed out in Section 1, unreasonable results of this type cannot in general be avoided, but it must be possible to find a better apportionment than that of Table 6.

## 4 Normalization by Canton or by Constituent People

Table 2 shows that the row and column sums of the $q_{i,j}$-matrix differ significantly from $r_i$ and $s_j$. Therefore, it is not surprising that proportional distribution based on $p_{i,j}$ or $q_{i,j}$ soon leads to some of the constraints (3) and (4) becoming binding, after which strange things can happen. In the example, condition (4) plays a more important role than (3); see Table 4 and the description in Section 3.

Can the problem be solved, at least in part, by scaling the numbers $q_{i,j}$ of each row up or down so that they get the "correct" sum, or by doing the same for the columns? Both possibilities will be considered.

Concerning the rows, this idea is realized by defining, for all $i$ and $j$:

(5) $$q_{i,j}^C = p_{i,j} \cdot r_i / p_{i,N}$$

Note that $p_{i,N}$ is the total population of Canton $i$. The definition implies

$$q_{i,N}^C = r_i \text{ for all } i$$

Therefore, the numbers $q_{i,j}^C$ are exactly the result of scaling $q_{i,j}$ up or down so that the sum of row $i$ equals $r_i$. This process could be called "normalization by Canton"; hence the

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11. Cantons 8 and 10 have already gotten the seats to which they are entitled by condition (3).

12. When $I - 1$ of the equalities (3) or $J - 1$ of the equalities (4) are satisfied, the rest of the allocation is determined by the constraints. This must hold when $H - 1$ seats have been handed out, but the example shows that it can happen much earlier.

superscript *C*.

The $q_{i,j}^C$-matrix is shown in Table 7. The row sums equal $r_i$, but the column sums deviate from $s_j$.

The seats can be distributed by the method described in Section 3, with the point of departure being $q_{i,j}^C$ instead of $p_{i,j}$ or $q_{i,j}$.

In this case, 59 seats are distributed without the divisor method having to be modified, and 69 seats are distributed before it becomes pointless to compute more quotients.[13] That is, the latter part of the procedure, where everything is determined by (3) and (4), involves *eleven* seats. The final apportionment is shown in Table 8.

Instead of adjusting each row in the $q_{i,j}$-matrix so that the row sums equal $r_i$, one can treat the columns in a similar manner. This is achieved by defining, for each *i* and *j*:

(6) $$q_{i,j}^P = p_{i,j} \cdot s_j / p_{M,j}$$

Here $p_{M,j}$ is the total population of Constituent People *j*. The definition implies

$$q_{M,j}^P = s_j \text{ for all } j$$

This process could be called "normalization by Constituent People"; hence the superscript *P*. The numbers $q_{i,j}^P$ are shown in Table 9. Apart from an unimportant round-off error, the column sums equal $s_j$, but the row sums deviate from $r_i$.

Table 10 gives the result of basing the apportionment on $q_{i,j}^P$. In this case, 63 seats are distributed without the divisor method having to be modified, and 77 seats are handed out before it becomes pointless to compute more quotients.[14] Hence only *three* seats are involved in the part of the process where everything is determined by (3) and (4).

From a theoretical point of view, both procedures considered here must be considered more satisfactory than the one presented in Section 3. Which of the two is the better one?

In general, one would expect that proportionality is best achieved if the constraints become binding late in the process and few seats are distributed solely on the basis of the constraints. In the example, normalization by Constituent People works better than normalization by Canton by this criterion. In addition, it is condition (4) that actually becomes binding in the original distribution of Section 3, which seems to indicate that there is more to gain by normalization by Constituent People than by normalization by Canton.

On the other hand, the apportionment of Table 8 intuitively appears more proportional to population than that of Table 10.[15]

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13. Detailed information about the distribution, corresponding to Tables 3 – 6, is available, but is not included here. — When the 58th seat has been handed out, Canton 5 has gotten the number of seats to which it is entitled by condition (3). Similarly, when the 59th seat has been awarded, Constituent People 1 (Bosniacs) has gotten the number of seats to which it is entitled by condition (4). The 60th quotient also belongs to a Bosniac group and can therefore not be used to distribute a seat. The 76th quotient is used to hand out the 69th seat. Then four of the ten Cantons and two of the three Constituent Peoples have gotten the number of seats to which they are entitled.

14. Again, information corresponding to Tables 3 – 6 is available but not included here. — When the 42nd seat has been awarded, Canton 10 has gotten the number of seats to which it is entitled by condition (3). Nevertheless, the odd numbers method can be applied without modifications up to and including the 63rd seat. The 86th quotient is used to hand out the 77th seat. Then eight Cantons and two Constituent Peoples have gotten the number of seats to which they are entitled.

15. In particular, normalization by Canton leaves group C 5 without any seat, which is certainly the reasonable

(continues…)

All in all, there do not seem to be strong reasons for preferring normalizing by Canton to normalizing by Constituent People, or vice versa.

Is it possible to do better than both these alternatives, by in a sense normalizing along both dimensions? The answer is yes, as explained in Sections 5 and 6.

## 5 The continuous case

Although the problem presented in Section 2 is of the discrete type, it is convenient to return to the continuous case, described in Section 1.

The following statement is a consequence of [1] Theorem 1 (page 704):[16]

> There exist positive numbers $\delta$, $\lambda_i$, and $\mu_j$ so that (1) and (2) are satisfied when $a_{i,j}$ is defined by
>
> (7) $$a_{i,j} = \delta \cdot \lambda_i \cdot \mu_j \cdot p_{i,j}$$
>
> Moreover, the numbers $a_{i,j}$ are uniquely determined by the requirement that they be of the form (7) and satisfy (1) and (2).

The numbers $\delta$, $\lambda_i$, and $\mu_j$ are not uniquely determined. For example, nothing changes if $\delta$ is multiplied by a positive constant and each $\lambda_i$, or each $\mu_j$, is divided by the same constant.[17]

It is convenient to choose $\delta$ equal to the "normalization factor" $H / p_{M,N}$. Then (7) becomes:

(8) $$a_{i,j} = \lambda_i \cdot \mu_j \cdot q_{i,j}$$

Even here $\lambda_i$ and $\mu_j$ are not unique.

In [1], the theorem is proved by a fixed-point argument.[18]

The $a_{i,j}$-matrix is obtained by scaling the rows and columns of the $q_{i,j}$-matrix up or down to the extent necessary to satisfy (1) and (2). Apart from this, $a_{i,j}$ is equal to $q_{i,j}$. Therefore, it can reasonably be claimed that the numbers $a_{i,j}$, given by (7) or (8), are the solution to the

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(…continued)
outcome, while normalization by Constituent People gives this group two seats, as did the procedure of Section 3. — Tables 6 and 10 are almost equal. Groups O 3 and C 6 get one seat more in Table 10 than in Table 6, while C 3 and O 6 get one seat less. These are the only differences. The two apportionments are as close as they can be when they are not identical. It is easy to prove that if two apportionments satisfying (3) and (4) are not equal, they must differ for at least four entities.

16. In the most general formulation of [1], the equality constraints (1) and (2) are replaced by upper and lower bounds on $a_{i,N}$ and $a_{M,j}$. It is not required that *all* the numbers $p_{i,j}$ be strictly positive, although they must all be non-negative, and there cannot be any row or column in the $p_{i,j}$-matrix consisting entirely of zeros. These generalizations make things more complicated and are not considered here.

17. As the statement is formulated, it is required that $\delta$, $\lambda_i$, and $\mu_j$ be strictly positive. Actually, it is sufficient to assume that they are non-negative. It can then be proved that they are strictly positive. Suppose, for example, that $\lambda_i = 0$ for some $i$. Then (7) implies $a_{i,j} = 0$ for this $i$ and all $j$. Therefore, (1) gives $r_i = a_{i,N} = 0$, contradicting the assumption that $r_i$ is strictly positive.

18. The algorithm outlined below can also serve as a proof of the statement.

continuous problem. They are as far as possible proportional to $q_{i,j}$, and hence to $p_{i,j}$, only with the deviations necessary to satisfy (1) and (2).[19] They represent "normalizing along both dimensions"; see the question asked at the end of Section 4.

The numbers $a_{i,j}$ can be computed by an iteration algorithm; see [2] Section 1.4. A sketch of the algorithm is given here:[20]

The computation is carried out in *stages*, starting with stage 0. The variable $k$ is used to denote the stages. Hence $k = 0, 1, 2, \ldots$ . At stage $k$, positive real numbers $\lambda_i^{(k)}$ and $\mu_j^{(k)}$ are computed and $q_{i,j}^{(k)}$ is defined by:

(9) $$q_{i,j}^{(k)} = \lambda_i^{(k)} \cdot \mu_j^{(k)} \cdot q_{i,j}$$

The goal is to have the $q_{i,j}^{(k)}$-matrix satisfy analogies of (1) and (2), that is

(10) $$q_{i,N}^{(k)} = r_i \text{ for all } i$$

(11) $$q_{M,j}^{(k)} = s_j \text{ for all } j$$

In general, this will never be achieved exactly. It is necessary to keep track of the aggregate deviation from (10) and (11) at each stage. This motivates the following definitions:

$$\varepsilon_1^{(k)} = \Sigma_{i \in M} \, | q_{i,N}^{(k)} - r_i |$$

$$\varepsilon_2^{(k)} = \Sigma_{j \in N} \, | q_{M,j}^{(k)} - s_j |$$

The *total error* at stage $k$ is given by $\varepsilon^{(k)} = \varepsilon_1^{(k)} + \varepsilon_2^{(k)}$.

Initially, $\lambda_i^{(0)} = \mu_j^{(0)} = 1$, which gives $q_{i,j}^{(0)} = q_{i,j}$. Neither (10) nor (11) can be expected to hold.

If $k > 0$ is an odd integer, $q_{i,j}^{(k)}$ is computed from $q_{i,j}^{(k-1)}$ by a procedure of "normalization by row", in analogy with (5). This is achieved by defining:

$$\lambda_i^{(k)} = r_i \,/\, \Sigma_{j \in N} \, \mu_j^{(k-1)} \cdot q_{i,j}$$

$$\mu_j^{(k)} = \mu_j^{(k-1)}$$

When $q_{i,j}^{(k)}$ is defined by (9), (10) holds. That is, $\varepsilon_1^{(k)} = 0$, which implies $\varepsilon^{(k)} = \varepsilon_2^{(k)}$. In general, $\varepsilon^{(k)} > 0$.

If $k > 0$ is even, $q_{i,j}^{(k)}$ is computed from $q_{i,j}^{(k-1)}$ by "normalization by column", in analogy with (6). This amounts to defining:

$$\lambda_i^{(k)} = \lambda_i^{(k-1)}$$

$$\mu_j^{(k)} = s_j \,/\, \Sigma_{i \in M} \, \lambda_i^{(k-1)} \cdot q_{i,j}$$

When $q_{i,j}^{(k)}$ is again defined by (9), (11) holds and $\varepsilon_2^{(k)} = 0$. Then $\varepsilon^{(k)} = \varepsilon_1^{(k)}$, which generally is positive.

It can be proved that the procedure converges. In particular, $\lim_{k\to\infty} \varepsilon^{(k)} = 0$. Let

$$\lambda_i = \lim_{k\to\infty} \lambda_i^{(k)}$$

$$\mu_j = \lim_{k\to\infty} \mu_j^{(k)}$$

$$a_{i,j} = \lim_{k\to\infty} q_{i,j}^{(k)} = \lambda_i \cdot \mu_j \cdot q_{i,j}$$

Then the numbers $a_{i,j}$ are of the form (8) and satisfy (1) and (2).

Of course, it is impossible to complete infinitely many stages of the algorithm. In

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19. Moreover, the procedure described here is the *only* solution to the continuous problem that satisfies a set of reasonable axioms; see [1] Theorem 2 (page 705).

20. The algorithm of [2] is designed to take care of the general case considered in [1]; see note 16. Here it has been simplified to fit the problem considered.

practice, it is terminated when $\varepsilon^{(k-1)} + \varepsilon^{(k)} < \varepsilon$ for some predetermined (small) number $\varepsilon > 0$. It is necessary to use the sum of the errors at two consecutive stages in order to get non-trivial cases of deviation from both (10) and (11).[21] The result is given by $a_{i,j} = q_{i,j}^{(k)}$, $\lambda_i = \lambda_i^{(k)}$, and $\mu_j = \mu_j^{(k)}$, for the $k$ at which the computation stops.

The results of applying this procedure to the example are given in Table 11.[22] The $a_{i,j}$-matrix is shown, as well as the numbers $\lambda_i$ and $\mu_j$. The latter are denoted $\lambda_i^c$ and $\mu_j^c$, where the superscript $c$ indicates that this is the continuous case. They give an impression of how the rows and columns of the $q_{i,j}$-matrix are scaled up or down to get the $a_{i,j}$-matrix. Therefore, $\lambda_i^c$ and $\mu_j^c$ can be said to determine the two-dimensional normalization.

## 6 Apportionment based on the continuous allocation

Although the continuous problem is now solved, the discrete one is not. The numbers $a_{i,j}$ of Table 11 are not integers and cannot directly be used to distribute the seats in the House of Peoples, but they can serve as a basis for the computation.

The seats can be distributed by the method described in Section 3, with the point of departure being $a_{i,j}$ instead of $p_{i,j}$ or $q_{i,j}$.

The distribution is documented in Tables 12 and 13. They contain information similar to that of Tables 3 and 4; see the description of these tables in Section 3.

In this case, it is possible to distribute 78 of the 80 seats on the basis of the largest quotients, without any Canton or Constituent People getting more seats than permitted by (3) or (4). That is, up to and including the 78th seats the odd numbers method can be applied without modifications. However, quotients nos. 79 – 82 cannot be used to distribute seats.

The 83rd quotient is used to hand out the 79th seat. The final apportionment is now determined, and further computing and comparing of quotients is pointless. It follows from (3) that the 80th seat must be given to Canton 2, and it follows from (4) that it must be given to Constituent People 3 (Others).[23] Table 14 shows the distribution of the 79 first seats, and Table 15 gives the final apportionment.[24]

---

21. The convergence seems to be fairly rapid. The computation in the example was carried out in a few seconds on a desk-top computer. The chosen value of $\varepsilon$ was $10^{-12}$.

22. The numbers are given to an accuracy of four digits after the decimal point, but the computation has been carried out with higher accuracy. Compared to $r_i$ and $s_j$ there seems to be a deviation of 0.0001 in some of the row and column sums. This is due to accumulation of round-off errors when the numbers are rounded to four digits.

23. If one goes on computing quotients, it will last until the 134th quotient before the last seat is awarded.

24. After 78 seats have been awarded, the constraints permit only two possibilities. The two remaining seats must be given to groups B 6 and O 2, as actually happened, or to groups B 2 and O 6. — If 80 seats are distributed by the odd numbers method, Canton 1 will get one seat more and Canton 6 one seat less than prescribed by (3), and Constituent People 2 (Croats) will get one seat more and Constituent People 3 (Others) one seat less than prescribed by (4). These are the only deviations from (3) and (4).

## 7 Direct solution of the discrete case – theory

The method presented in Section 6 seems reasonable, and in the example it gives an intuitively acceptable result. It is reassuring that as many as 78 of the 80 seats are distributed without the divisor method having to be modified.

It does not follow that all problems are solved.

There will always be a phase where everything is determined by the constraints. In the example, this only involves *one* seat, but the number could easily be higher. Perhaps it is more relevant to concentrate on the seats that are not handed out by the unmodified divisor method. In the example, there are *two* such seats. When these seats are awarded, one can say that the method of proportional distribution has lost control and the constraints have taken over. This creates a danger of unreasonable results.[25] As pointed out in Section 1, it may be impossible to find an apportionment which is "reasonable" in all relevant respects, but still it seems unsatisfactory that the same principles cannot be applied during the whole process of distribution. There are reasons to suspect that the final apportionment could deviates more from proportionality than necessary to satisfy the constraints (3) and (4) and the requirement that each $h_{i,j}$ be a non-negative integer.

In Section 5, the rows and columns of the $q_{i,j}$-matrix are adjusted so as to obtain the row and column sums prescribed by (1) and (2). This form of adjustment is well suited for the continuous problem.

In the discrete case, however, another type of adjustment seems more appropriate, namely the following: Adjust the rows and columns in such a way that, when the adjusted numbers are used to distribute $H$ seats by the odd numbers method, the apportionment satisfies (3) and (4). During the distribution of the $H$ seats, the odd numbers method shall be applied without modifications, although ties may be broken in the way most favorable to the fulfillment of (3) and (4). The initial adjustment of the $q_{i,j}$-matrix shall guarantee that the constraints can be satisfied.

Is it possible to make such adjustments? The answer is yes. The following is a consequence of [1] Theorem 5 (page 712):[26]

> There exist positive real numbers $\delta$, $\lambda_i$, and $\mu_j$ with the following property: Let $b_{i,j}$ be defined by
>
> (12) $$b_{i,j} = \delta \cdot \lambda_i \cdot \mu_j \cdot q_{i,j}$$
>
> Distribute $H$ seats according to the odd numbers method on the basis of $b_{i,j}$. Then the apportionment, or at least one of the possible apportionments in case of a tie, satisfies (3) and (4).

---

25. In the example, the procedure of Section 3 gives two seats to the tiny group C 5, as does normalization by Constituent People as described in Section 4. Although nothing like this happens for the procedure of Section 6, one cannot *a priori* rule out the possibility of similar effects.

26. A similar statement holds for each divisor method in the class described in note 39. For reasons explained in Section 3, the odd numbers method is preferred here.

The use of $q_{i,j}$ instead of $p_{i,j}$ in (12) is a matter of convenience and makes no difference, since the distribution is unaffected by the numbers $b_{i,j}$ being multiplied by a common, positive factor. For example, $\delta$ can be chosen so that $b_{M,N} = H$. The numbers $b_{i,j}$ are usually not unique, not even up to a common factor. The *apportionment* is, however, essentially unique. It is unique except possibly when ties occur in the application of the odd numbers method.[27]

In general, it is not possible to choose $\delta$, $\lambda_i$, and $\mu_j$ in (12) so that the $b_{i,j}$-matrix is an allocation satisfying (1) and (2).

For a real number $x > 0$, the symbol $[x]_{½}$ is used to denote a set which contains one or two integers. If $x = k + ½$ for an integer $k$, $[x]_{½} = \{k, k+1\}$. Otherwise, $[x]_{½}$ has one element, namely the result of rounding $x$ up or down to the nearest integer. In any case, $k \in [x]_{½}$ if and only if $k$ is an integer and $k - ½ \le x \le k + ½$.[28]

General properties of the odd numbers method imply that the apportionment given by (12) can also be characterized as follows:[29]

> There exist positive real numbers $\delta$, $\lambda_i$, and $\mu_j$ and non-negative integers $h_{i,j}$ with the following property: When $b_{i,j}$ is defined by $b_{i,j} = \delta \cdot \lambda_i \cdot \mu_j \cdot q_{i,j}$, then
>
> (13) $$h_{i,j} \in [b_{i,j}]_{½}$$
>
> for all $i$ and $j$, and the $h_{i,j}$-matrix is an apportionment satisfying (3) and (4).

The apportionment is essentially unique. A necessary, but not sufficient, condition for it not being unique, is that some of the sets $[b_{i,j}]_{½}$ contain two elements, that is, some of the numbers $b_{i,j}$ lie midway between two integers.[30]

Only the relative size of the numbers $b_{i,j}$ plays a role in (12), but in (13) their absolute size is of importance. Any parameters $\delta$, $\lambda_i$, and $\mu_j$ that can be used in (13), can also be used in (12), but not vice versa. It is not necessarily possible to have $b_{M,N} = H$ in (13). The parameter $\delta$ is superfluous in (13) as well as in (12); it can, for example, be incorporated into $\lambda_i$.

---

27. The potential non-uniqueness is unavoidable. In practice, however, the probability that the apportionment is unique, is very close to 1. See further discussion in note 30.

28. The definition also makes sense when $x \le 0$, but that case has no interest in the present connection. A generalization of the definition is given in the Appendix, note 39.

29. This is the formulation used in [1]. – The equivalence of the characterizations given by (12) and (13) follows from the argument used in the Appendix to prove that the odd numbers method can be characterized by (A3).

30. To be precise, the following can be proved (see [1] Lemmas 2 and 5): Let two apportionments satisfying (3) and (4) be given by $h_{i,j}$ and $h_{i,j}'$, and assume that they both are of the form (13). That is, there exist positive numbers $\delta$, $\lambda_i$, $\mu_j$, $b_{i,j}$, $\delta'$, $\lambda_i'$, $\mu_j'$, and $b_{i,j}'$ such that $b_{i,j} = \delta \cdot \lambda_i \cdot \mu_j \cdot q_{i,j}$, $b_{i,j}' = \delta' \cdot \lambda_i' \cdot \mu_j' \cdot q_{i,j}$, $h_{i,j} \in [b_{i,j}]_{½}$, and $h_{i,j}' \in [b_{i,j}']_{½}$ for all $i$ and $j$. Suppose $h_{i,j} \ne h_{i,j}'$ for some $i$ and $j$. Then $b_{i,j} = b_{i,j}'$. This is only possible if $|h_{i,j} - h_{i,j}'| = 1$ and $b_{i,j} = b_{i,j}'$ lies midway between $h_{i,j}$ and $h_{i,j}'$. — If the apportionments given by $h_{i,j}$ and $h_{i,j}'$ are not equal, the constraints imply that they must differ for at least *four* entities. That is, at least four of the numbers $b_{i,j}$ must be of the form $k + ½$ for an integer $k$. Although $\delta$, $\lambda_i$ and $\mu_j$ are endogenous, this is essentially impossible when the original data, $p_{i,j}$, are naturally generated. — The one-dimensional, and therefore considerably simpler, version of this uniqueness result is stated and proved in note 38.

The existence of an apportionment of the form (12) or (13) is proved in [2] Section 2.2. The proof is constructive, that is, an algorithm for computing the apportionment is presented. It is fairly complicated. An outline is given here.[31]

The algorithm starts with an apportionment which is of the form (13) and distributes the correct number of seats. That is, there are given positive real numbers $\delta$, $\lambda_i$, and $\mu_j$ and non-negative integers $h_{i,j}$ so that (13) holds for $b_{i,j} = \delta \cdot \lambda_i \cdot \mu_j \cdot q_{i,j}$, and $h_{M,N} = H$. In general, (3) and (4) are not satisfied. Any apportionment which is of the form (13) and satisfies $h_{M,N} = H$ can be used as a starting point, but it is an advantage that it comes as close as possible to fulfilling (3) and (4).

It seems convenient to use the numbers $a_{i,j}$, $\lambda_i$, and $\mu_j$ computed in Section 5 as the point of departure. Let $h_{i,j}$ be the result of distributing $H$ seats by the odd numbers method on the basis of $a_{i,j}$. Then there exists a number $\delta > 0$ such that $h_{i,j} \in [\delta \cdot a_{i,j}]_{½}$ for all $i$ and $j$.[32] Let $b_{i,j} = \delta \cdot a_{i,j}$, so that

(14) $$h_{i,j} \in [b_{i,j}]_{½}$$

By (8), $b_{i,j} = \delta \cdot \lambda_i \cdot \mu_j \cdot q_{i,j}$. Hence an apportionment of the form (13) satisfying $h_{M,N} = H$ has been found.[33]

The total deviations from (3) and (4), respectively, are measured as follows:

$$E_1 = \Sigma_{i \in M} \; | h_{i,N} - r_i |$$
$$E_2 = \Sigma_{j \in N} \; | h_{M,j} - s_j |$$

Since $h_{M,N} = H = r_M = s_N$, $E_1$ and $E_2$ are even integers. They are obviously non-negative. All equalities in (3) are satisfied if and only if $E_1 = 0$, and all equalities in (4) are satisfied if and only if $E_2 = 0$.

If $E_1 = E_2 = 0$, a solution has been found and the computation is completed. Assume $E_1 > 0$. (The case of $E_2 > 0$ is similar.) Since $h_{M,N} = H = r_M$, there must be deviations from (3) in both directions. Choose $i$ and $i'$ such that $h_{i,N} < r_i$ and $h_{i',N} > r_{i'}$. Then it seems reasonable to increase $\lambda_i$ and decrease $\lambda_{i'}$, or at least increase $\lambda_i / \lambda_{i'}$, in order to increase $h_{i,N}$ and decrease $h_{i',N}$. This is the general idea of the algorithm. It is not quite as simple, however. One must also take account of the columns. Even if an adjustment of the $h_{i,j}$-matrix brings $h_{i,N}$ closer to $r_i$ and $h_{i',N}$ closer to $r_{i'}$, so that $E_1$ decreases, little is gained if new or more severe deviations from (4) emerge, so that $E_2$ increases.

The computation is carried out in *steps*. At any point in time, numbers $\delta$, $\lambda_i$, $\mu_j$, $b_{i,j}$, and

---

31. The problem considered in [2] is more general than the one discussed here. First, there are the generalizations mentioned in note 16. Second, other divisor methods than the odd numbers method can be used. The outline of the algorithm in the main text is simplified and adopted to the special case considered. – In [1] Theorem 6 (page 713) a characterization result is presented, somewhat analogous to the one mentioned in note 19. In this case, however, the axioms are not satisfied by only *one* procedure, but by one procedure for each divisor method in the class described in note 39.

32. See the discussion of (A3) in the Appendix.

33. For the example, the relevant data are given in Table 11. The result of distributing $H = 80$ seats by the odd numbers method on the basis of $a_{i,j}$ can be found from Table 13. The deviations from (3) and (4) are mentioned in note 24. The 80th quotient is 0.5076 > ½ and the 81st quotient is 0.4942 < ½. Hence $\delta$ can be chosen equal to 1. The method of the largest remainder gives the same result as the odd numbers method in this case. These claims follow from arguments made in the Appendix.

$h_{i,j}$ are given. Here $\delta$, $\lambda_i$, $\mu_j$, and $b_{i,j}$ are positive real number, $h_{i,j}$ is a non-negative integer, $b_{i,j} = \delta\cdot\lambda_i\cdot\mu_j\cdot q_{i,j}$, and $h_{i,j} \in [b_{i,j}]_{½}$.

Each step starts with a process in which certain rows and columns are *labeled*, while certain entities are *marked*, either *positively* or *negatively*:

(a) If $h_{i,N} < r_i$, then row *i* is labeled.

(b) If row *i* is labeled and column *j* is not, and $b_{i,j} = h_{i,j} + ½$, then column *j* is labeled, and entity $(i, j)$ is positively marked.

(c) If column *j* is labeled and row *i* is not, and $b_{i,j} = h_{i,j} – ½$ (which is possible only if $h_{i,j} \geq 1$), then row *i* is labeled, and entity $(i, j)$ is negatively marked.

This process is repeated until no further labeling is possible. Two cases are distinguished:

(i) There exists no *i* such that row *i* is labeled and $h_{i,N} > r_i$.

(ii) There exists an *i* such that row *i* is labeled and $h_{i,N} > r_i$.

In case (i), some of the numbers $\lambda_i$ and $\mu_j$, and hence some $b_{i,j} = \delta\cdot\lambda_i\cdot\mu_j\cdot q_{i,j}$, are changed, but all $h_{i,j}$ are kept constant. In case (ii), some of the numbers $h_{i,j}$ are changed but all $b_{i,j}$ are kept constant. In both cases, $h_{i,j} \in [b_{i,j}]_{½}$ shall remain true for all *i* and *j*.

Case (i) is considered first. Assume, therefore, that the labeling process is terminated without any row *i* with $h_{i,N} > r_i$ having been labeled.

Let $A_1$ be the set of entities $(i, j)$ such that *i* is labeled and *j* is not labeled, and let $A_2$ be the set of entities $(i, j)$ such that *j* is labeled, *i* is not labeled, and $h_{i,j} \geq 1$. It is obvious that $A_1$ and $A_2$ are disjoint. Suppose they are both empty. Since $E_1 > 0$, there exist *i* and *i'* such that $h_{i,N} < r_i$ and $h_{i',N} > r_{i'}$. Row *i* is labeled by (a) of the labeling process. Row *i'* is not labeled; otherwise, case (ii) would apply. Since $A_1$ is empty and *i* is labeled, all columns are labeled. Since $A_2$ is empty and *i'* is not labeled, $h_{i',j} = 0$ for all *j*, implying $h_{i',N} = 0$, which contradicts $h_{i',N} > r_{i'}$. Therefore, the assumption that $A_1$ and $A_2$ are both empty, must be wrong.

For each $(i, j) \in A_1$, define $\varepsilon_{i,j} = (h_{i,j}+½) / b_{i,j}$. Since $h_{i,j} \in [b_{i,j}]_{½}$, $b_{i,j} \leq h_{i,j} + ½$. If $b_{i,j} = h_{i,j} + ½$, $(i, j) \in A_1$ would contradict (b) of the labeling process. Since $b_{i,j} > 0$, this gives $\varepsilon_{i,j} > 1$. For each $(i, j) \in A_2$, define $\varepsilon_{i,j} = b_{i,j} / (h_{i,j}–½)$. The definition of $A_2$ implies $h_{i,j} \geq 1$, which gives $h_{i,j} – ½ > 0$. Since $h_{i,j} \in [b_{i,j}]_{½}$, $b_{i,j} \geq h_{i,j} – ½$. If $b_{i,j} = h_{i,j} – ½$, $(i, j) \in A_2$ would contradict (c) of the labeling process. Therefore, $\varepsilon_{i,j} > 1$ in this case as well. Because $A_1$ and $A_2$ are disjoint, no number $\varepsilon_{i,j}$ has been given two definitions. Let $\varepsilon$ be the smallest of the numbers $\varepsilon_{i,j}$ for $(i, j) \in A_1 \cup A_2$. Since $A_1$ and $A_2$ are not both empty, this is well defined and $\varepsilon > 1$.

Let $\lambda_i' = \varepsilon \cdot \lambda_i$ if *i* is labeled, $\lambda_i' = \lambda_i$ if *i* is not labeled, $\mu_j' = \mu_j / \varepsilon$ if *j* is labeled, and $\mu_j' = \mu_j$ if *j* is not labeled. Then define $b_{i,j}' = \delta\cdot\lambda_i'\cdot\mu_j'\cdot q_{i,j}$. If both *i* and *j* are labeled or none of them is, then $b_{i,j}' = b_{i,j}$; if *i* is labeled and *j* is not, then $b_{i,j}' > b_{i,j}$; and if *j* is labeled and *i* is not, then $b_{i,j}' < b_{i,j}$. If entity $(i, j)$ is marked, row *i* and column *j* are both labeled, and $b_{i,j}' = b_{i,j}$. The choice of $\varepsilon$ guarantees $h_{i,j} \in [b_{i,j}']_{½}$ for all *i* and *j*.

Now a new step is started, with $b_{i,j}'$ substituted for $b_{i,j}$. The labeling process will resemble, but not be equal to, that of the previous step. Every row or column that was labeled last time, is labeled again. In addition, at least one more row or column is labeled. If $\varepsilon = \varepsilon_{i,j}$ for $(i, j) \in A_1$, column *j* was not labeled at the previous step, but is labeled now. If $\varepsilon = \varepsilon_{i,j}$ for $(i, j) \in A_2$, the same holds for row *i*.

After a finite number of steps where case (i) applies, case (ii) will apply. At the latest this will happen after $I + J$ steps, since all rows and columns must then be labeled.

Then consider case (ii). The assumption $E_1 > 0$, (a) of the labeling process, and the case assumption, imply the existence of labeled rows *i* and *i'* so that $h_{i,N} < r_i$ and $h_{i',N} > r_{i'}$. By following the labeling backwards from $i'$, it is possible to find an integer $k \geq 2$ and rows $i_1$, $i_2$, ... , $i_k$ and columns $j_1, j_2, \ldots, j_{k-1}$ such that $i_1 = i$, $i_k = i'$, and:

– Rows $i_1, i_2, \ldots, i_k$ are labeled.
– Columns $j_1, j_2, \ldots, j_{k-1}$ are labeled.
– Entities $(i_1, j_1), (i_2, j_2), \ldots, (i_{k-1}, j_{k-1})$ are positively marked.
– Entities $(i_2, j_1), (i_3, j_2), \ldots, (i_k, j_{k-1})$ are negatively marked.

Modify the $h_{i,j}$-matrix by giving one more seat to each of the $k-1$ entities $(i_1, j_1)$, $(i_2, j_2)$, ... , and $(i_{k-1}, j_{k-1})$ and taking one seat away from each of $(i_2, j_1)$, $(i_3, j_2)$, ... , and $(i_k, j_{k-1})$. By the conditions for an entity being positively or negatively marked, this is possible, and $h_{i,j} \in [b_{i,j}]_{½}$ still holds. In the $h_{i,j}$-matrix, every column sum remains unchanged. The same is true for the row sums, except for rows $i$ and $i'$. Moreover, $h_{i,N}$ is increased and $h_{i',N}$ is decreased. Hence $E_1$ is decreased and $E_2$ is unchanged.

If $E_1$ is still positive, the procedure is repeated. The labeling and marking process will now be different, but after a finite number of steps at which case (i) applies, case (ii) will again apply, and $E_1$ is further decreased. After a finite number of steps, $E_1 = 0$ and (3) is satisfied for all $i$.

If $E_2 > 0$, the procedure is started over again with rows and columns interchanged. At this stage, the row sums do not change and $E_1 = 0$ remains true. After a finite number of steps, $E_2 = 0$, and a solution satisfying (3) and (4) has been found.

## 8 Direct solution of the discrete case – application

The result of applying this algorithm of Section 7 to the example is documented in Tables 16 and 17. The final apportionment, computed by the algorithm, is given in Table 17. The numbers in Table 16 are not, however, those that come out of the algorithm. Instead, they are chosen so that they are close to the numbers $a_{i,j}$ of Table 11.

Previous statements about uniqueness imply the following:[34] Let $b_{i,j} = \delta \cdot \lambda_i \cdot \mu_j \cdot q_{i,j}$ for positive real numbers $\delta$, $\lambda_i$, and $\mu_j$, and assume that $h_{i,j}$ is the unique result of rounding $b_{i,j}$ up or down to the nearest integer. That is, no $h_{i,j}$ lies midway between two integers. If the $h_{i,j}$-matrix is an apportionment satisfying (3) or (4), it is the unique solution to the problem.

The real numbers $b_{i,j}$ of Table 16 are of the required form, and they relate to the integers $h_{i,j}$ of Table 17 in the correct way. Hence they prove that the solution has been found. In this case, $\delta = 1$, while $\lambda_i$ and $\mu_j$ are given in the last column and the last row, respectively, of Table 16. They are denoted $\lambda_i^d$ and $\mu_j^d$, where the superscript $d$ indicates that this is the discrete case. They give an impression of how the rows and columns of the $q_{i,j}$-matrix are scaled up or down to get the $b_{i,j}$-matrix.[35]

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34. See, in particular, note 30.

35. Compared to Table 11, $\lambda_1$ is decreased so as to bring entry (1, 1) slightly below 5.5, and $\lambda_6$ is increased so as to bring entry (6, 1) slightly above 2.5, implying that these entries will be rounded to 5 and 3, respectively. Similarly, $\mu_2$ is decreased and $\mu_3$ increased to get entry (7, 2) below 5.5 and entry (7, 3) above 1.5, so that the entries are rounded to 5 and 2. The reduction in $\mu_2$ has brought entry (2, 2) below 2.5, and this must be compensated by an increase in $\lambda_2$. No further adjustments are necessary. That is, $\mu_1^d = \mu_1^c$, and $\lambda_i^d = \lambda_i^c$ except for $i$ = 1, 2, and 6.

## 9 Summary

Table 18 summarizes the apportionments obtained by the five methods considered in Sections 3 – 8.

A reasonable measure of the distance between the apportionments given by $h_{i,j}$ and $h_{i,j}'$ is $\frac{1}{2}\cdot\Sigma_{i\in M,j\in N}\,|h_{i,j} - h_{i,j}'|$. Table 19 gives the distance, measured in this way, between any two of the five apportionments.

## Appendix: The relationship between the odd numbers method and the method of the largest remainder

Summary The odd numbers method and the method of the largest remainder are described.
An example of the Alabama paradox is given.
The relationship between the two methods is investigated. Let $d^+$ be the smallest fractional part which is rounded upwards, and let $d^-$ be the largest fractional part which is rounded downwards, when seats are distributed by the method of the largest remainder. The following three cases are exhaustive and mutually exclusive:
(a) $d^- > ½$
(b) $d^- \leq ½ \leq d^+$
(c) $d^+ < ½$
In case (b), the two methods are equal. In case (a), they may be equal, but if they are different, the odd numbers method is most favorable to small entities. In case (c), any difference must go in the direction of the odd numbers method being most favorable to large entities.

Introduction The contents of this appendix is a digression compared to the discussion above. Here only one-dimensional discrete problems are considered. Therefore, the notation is somewhat different. When definitions are equal, they are repeated, so as to make the appendix self contained.

There are given $H$ units of some indivisible good. They are to be distributed among $I$ entities. Both $H$ and $I$ are positive integers. Everything is trivial if $I = 1$; hence $I \geq 2$ is assumed. The variable $i$ varies over the sets $M = \{1, 2, ... I\}$.

There are given positive real numbers $p_i$. They represent the entities' claims on a share of the $H$ units of the good. As far as possible the distribution should be proportional to the numbers $p_i$. Let

$$p_M = \Sigma_{i\in M}\, p_i$$

Similar notation is used for other indexed variables.

It is convenient to *normalize* the numbers $p_i$ by defining

$$q_i = p_i \cdot H / p_M \text{ for all } i$$

Then $q_M = H$. Below, the distribution is based on $q_i$ rather than $p_i$. This never makes any difference for the distribution as such, but it is essential for some of the statements made.

The most important applications of the theory deal with the distribution of seats in elected assemblies among geographical areas, political parties, or other groups. Hence the units of the indivisible good are usually called *seats*. The entities are the geographical areas, the parties, or the groups. The number $p_i$ could, for example, be the population of geographical unit $i$, or the number of votes cast for party $i$ at an election.

Let $h_i$ denote the number of seats given to entity $i$. These must be non-negative integers satisfying $h_M = \Sigma_{i\in M}\, h_i = H$. The $h_i$-vector is called an *apportionment*.

For a real number $x > 0$, $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$. That is, $\lfloor x \rfloor$ is $x$ rounded downwards to the nearest integer. The symbol $[x]_{½}$ is used to denote a set which contains one or two integers. If $x = k + ½$ for an integer $k$, $[x]_{½} = \{k, k+1\}$. Otherwise, $[x]_{½}$ has one element, namely the result of rounding $x$ up or down to the nearest integer. In any case, $k \in [x]_{½}$ if and only if $k$ is an integer and $k - ½ \leq x \leq k + ½$.

<u>The odd numbers method</u> Each $q_i$ is divided by the odd numbers, 1, 3,, 5, and so on, as far as necessary for distributing the *H* seats.[36] For given *i*, the quotients obviously form a decreasing series. All the quotients, regardless of the entity to which they belong, are ordered by size. (If several quotients are equal, they shall of course occupy the corresponding number of places in the ordering.) The entity to which the largest quotient belongs, is awarded the first seat, the second largest quotient is used to hand out the second seat, and so on until all the *H* seats are distributed. Ties are broken arbitrarily.

Let $c^+$ be the smallest quotient for which a seat is won, and let $c^-$ be the largest quotient for which no seat is won. That is, $c^+$ and $c^-$ are quotients nos. *H* and $H + 1$, respectively. Clearly, $c^+ \geq c^- > 0$. Unless there is a tie for the last seat, $c^+ > c^-$.

Entity *i* competes for its seat no. *k* on the basis of the quotient obtained by dividing $q_i$ by $2k - 1$. When the apportionment is given by the integers $h_i$, the description of the procedure implies

(A1) $$q_i / (2h_i - 1) \geq c^+ \geq c^- \geq q_i / (2h_i + 1)$$

for all *i*. It is always possible to find a number $\delta > 0$ such that $\delta c^+ \geq ½ \geq \delta c^-$. Then (A1) gives

$$\delta q_i / (2h_i - 1) \geq \delta c^+ \geq ½ \geq \delta c^- \geq \delta q_i / (2h_i + 1)$$

This implies

(A2) $$h_i - ½ \leq \delta q_i \leq h_i + ½$$

That is, when the $h_i$-vector is an apportionment obtained by the odd numbers method, there exists a number $\delta > 0$ such that, for all *i*

(A3) $$h_i \in [\delta q_i]_{½}$$

Conversely, suppose that the non-negative integers $h_i$ and the positive real number $\delta$ satisfy (A3), and assume $h_M = H$. It makes no difference whether seats are distributed on the basis of $q_i$ or $\delta q_i$. When the latter numbers are used, it is easy to see that application of the odd numbers method can give the apportionment $h_i$. Any quotient greater than ½ gives a seat; a quotients equal to ½, if there are any, may or may not give a seat; and a quotient less than ½ does not give a seat.

It has now been proved that the set of possible apportionments according to the odd numbers method is equal to the set of $h_i$-vectors that are of the form (A3) and satisfy $h_M = H$. This holds both when this set has one element, and when it has two or more elements. In the former case, there is not a tie for the last seats, and $c^+ > c^-$. Then there is an interval of possible values of $\delta$.[37] For any possible value of $\delta$, at most one of the numbers $\delta q_i$ lies midway between two integers, that is, at most one of the sets $[\delta q_i]_{½}$ contains two elements. In

36. The distribution is not changed if all divisors are multiplied by the same positive constant. Therefore, the odd numbers method could also be presented as the divisor methods with divisors 0.5, 1.5, 2.5, and so on, that is, divisor no. *k* is $k - ½$. (In this case, the name "the odd numbers method" would not make much sense. The name "the method of major fractions" is sometimes used. It is inspired by the characterization (A3), presented below in the main text.)

37. The argument leading from (A1) to (A2) requires that $\delta c^+ \geq ½ \geq \delta c^-$, which is equivalent to $1/(2c^+) \leq \delta \leq 1/(2c^-)$. Any $\delta$ in this interval can be used in (A3). It is easy to prove that the converse also holds: If the $h_i$-vector satisfies (A3) and $\delta < 1/(2c^+)$, then $h_M < H$; if (A3) is satisfied and $\delta > 1/(2c^-)$, then $h_M > H$.

the latter case, there is a tie and $c^+ = c^-$. Then δ is uniquely determined.[38] At least two of the numbers $\delta q_i$ lie midway between two integers.

In short, if the apportionment is unique, the number δ of (A3) is not, and vice versa.

Other divisor methods are characterized by different series of divisors.[39]

The method of the largest remainder For each $i$, let $r_i = \lfloor q_i \rfloor$ and $s_i = q_i - r_i$. Then $r_i$ is a non-negative integer, $s_i$ is a real number with $0 \leq s_i < 1$, and $r_M + s_M = H$. Let $R = r_M$ and $S = s_M$. These numbers are always integers. It is theoretically possible that all $q_i$ are integers, in which case $r_i = q_i$ for all $i$ and $S = 0$. Then perfect proportionality is achieved by letting $h_i = q_i$. Otherwise, the number of entities for which $0 < s_i < 1$ is at least $S + 1$. Find the $S$ largest of the numbers $s_i$. (As above, if several numbers are equal, they shall be counted the appropriate number of times.) For each of the corresponding entities, let $h_i = r_i + 1$. For the rest of the entities, let $h_i = r_i$. Ties are broken arbitrarily.

Let $d^+$ be the smallest of the numbers $s_i$ which gave $h_i = r_i + 1$, and let $d^-$ be the largest of the remaining numbers $s_i$. That is, $d^+$ and $d^-$ are nos. $S$ and $S + 1$ when the numbers $s_i$ are ordered by size. If all $q_i$ are integers, this definition of $d^+$ does not make sense, since there is no $i$ with $h_i = r_i + 1$. In this case, let $d^+ = 1$. The general definition gives $d^- = 0$. In all other cases, $1 > d^+ \geq d^- > 0$. There is a tie if and only if $d^+ = d^-$.

The Alabama paradox A paradoxical and undesirable property of the method of the largest remainder is the following: If the total number of seats to be distributed is *increased*, everything else being kept constant, some entities can get *fewer* seats than before.[40]

The claim is only that the phenomenon *can* occur. Therefore, *one* example is sufficient proof. The example given here is as simple as possible, in the sense that $I$ and $H$ are minimal. The effect cannot occur for $I = 2$ or $H \leq 2$.

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38. The argument leading from (A1) to (A2) permits only one value of δ, namely $1/(2c^+)$. Could (A3) nevertheless be satisfied, for at least one of the possible apportionments, by other values of δ? The answer is no. Proof: Because there is a tie, there exist two different apportionments. Let them be given by $h_i$ and $h_i'$. Since $h_M = h_M' = H$, there must be an entity which gets more seats in the former case than in the latter, and vice versa. To be specific, assume $h_1 > h_1'$ and $h_2 < h_2'$. By assumption, both apportionments are of the form (A3); that is, there exist positive numbers δ and δ′ so that $h_i \in [\delta q_i]_{½}$ and $h_i' \in [\delta' q_i]_{½}$ for all $i$. Then $h_1 > h_1'$ implies $\delta q_1 \geq \delta' q_1$, and since $q_1 > 0$ this gives $\delta \geq \delta'$. Similarly, $h_2 < h_2'$ implies $\delta \leq \delta'$, and $\delta = \delta'$ follows. — What has now been proved, is the one-dimensional version of the statement formulated but not proved in note 30.

39. Any strictly increasing series $d_1, d_2, \ldots$ of positive divisors defines a divisor method. In [1] and [2] the following restrictions are introduced, in order to rule out certain unreasonable results: For all $k$, $k - 1 \leq d_k \leq k$; there are no $k$ and $k'$ so that $d_k = k - 1$ and $d_{k'} = k'$. (On the other hand, $d_1 = 0$ is permitted in [1] and [2]. In some applications, this possibility should not be ruled out, but it is ignored here in order to simplify.) — Let a series of divisors $\underline{d} = d_1, d_2, \ldots$ be given and assume that it satisfies the conditions just mentioned. For convenience, define $d_0 = 0$. For a real number $x > 0$, a set $[x]_{\underline{d}}$ is defined by $k \in [x]_{\underline{d}}$ if and only if $k$ is a non-negative integer and $d_k \leq x \leq d_{k+1}$. The divisor method given by $\underline{d}$ can be characterized by an analogy of (A3), with $[\delta q_i]_{\underline{d}}$ instead of $[\delta q_i]_{½}$. When the odd numbers method is given by $d_k = k - ½$ (see note 36), the definition of $[x]_{\underline{d}}$ given here coincides with the definition of $[x]_{½}$ in the main text.

40. The phenomenon was first discovered around 1880 in the USA, when computations were made concerning the distribution of the seats in the House of Representatives among the states. It was observed that the State of Alabama would lose a seat if the total membership of the House was increased.

First, let $I = 3$, $H = 3$, $p_1 = 135$, $p_2 = 129$, and $p_3 = 36$. Then $p_M = 300$ and $q_1 = 1.35$, $q_2 = 1.29$, and $q_3 = 0.36$. This gives $r_1 = 1$, $r_2 = 1$, $r_3 = 0$, and $S = 1$. The largest of the numbers $s_i$ is $s_3 = 0.36$, implying $h_1 = 1$, $h_2 = 1$, and $h_3 = 1$. In this case, $d^+ = 0.36$ and $d^- = 0.35$.

Second, assume $H = 4$, with $I$ and the $p_i$-vector being unchanged. Then the "normalization factor" $H / p_M$ is 4/300, and $q_1 = 1.8$, $q_2 = 1.72$, and $q_3 = 0.48$, giving $r_1 = 1$, $r_2 = 1$, $r_3 = 0$, and $S = 2$. The two largest of the numbers $s_i$ are $s_1 = 0.8$ and $s_2 = 0.72$, implying $h_1 = 2$, $h_2 = 2$, and $h_3 = 0$. In this case, $d^+ = 0.72$ and $d^- = 0.48$.

That is, entity 3 has lost a seat when the only change in the data is an increase in the total number of seats to be distributed. Note that there are no ties involved.[41]

The phenomenon cannot occur for a divisor method.[42] Suppose that a distribution of $H$ seats has taken place. If $H + 1$ seats shall later be distributed on the basis of the same $p_i$-vector, it is possible to imitate the original distribution of the $H$ seats, and then give seat no. $H + 1$ to the entity, or one of the entities, which is next in line for receiving a seat.

When the odd numbers method is applied to the example presented above, the relevant quotients, computed on the basis of the numbers $p_i$, are:

– For entity 1: 135, 45, 27
– For entity 2: 129, 43
– For entity 3: 36

For $H = 3$, this gives $h_1 = 2$, $h_2 = 1$, and $h_3 = 0$. For $H = 4$, it gives $h_1 = 2$, $h_2 = 2$, and $h_3 = 0$.

The numbers $c^+$ and $c^-$ are defined by computing quotients based on $q_i$. For $H = 3$, this gives $c^+ = 0.45$ and $c^- = 0.43$. For $H = 4$, it gives $c^+ \approx 0.57$ and $c^- = 0.48$.

Compared to the method of the largest remainder, the apportionment is different for $H = 3$ and equal for $H = 4$.

The relationship between the methods Let $I$, $H$, and $p_i$ for $i \in M$ be given, and let $h_i^O$ and $h_i^L$ denote apportionments resulting from distributing the $H$ seats by the odd numbers method and the method of the largest remainder, respectively. Define the numbers $r_i$, $s_i$, $R$, $S$, $c^+$, $c^-$, $d^+$, and $d^-$ as before. Note that $R + S = H$.

Consider the quotients computed for entity $i$ when the seats are distributed by the odd numbers method. Let $k_i$ be the number of these quotients that are strictly greater than ½, and let $k_i'$ be the number of quotients that are greater than or equal to ½. Define $K = k_M = \Sigma_{i \in M} k_i$ and $K' = k_M'$. That is, $K$ is the total number of quotients that are strictly greater than ½ when all the entities are taken together, while $K'$ is the total number of quotients that are greater than or equal to ½.

If $r_i > 0$, quotient no. $r_i$ for entity $i$ is

41. It is trivial that a similar effect can occur when there are ties. Let $I = 3$, $H = 1$, and $p_1 = p_2 = p_3 > 0$. If equals are treated equally, any entity can get the one seat, for example, it can be given to entity 1. Then let $H = 2$, the rest of the data remaining unchanged. Again, if equals are treated equally, any two of the entities can get one each of the two seats, for example, the seats can be given to entities 2 and 3. Entity 1 has lost a seat from the first to the second case. However, entity 1 *could* have gotten one seat in the second case. This distinguishes the non-trivial example of the main text from the trivial one given here.

42. Except in the trivial sense mentioned in note 41.

$$q_i / (2r_i–1) \geq r_i / (2r_i–1) > ½$$

Hence $k_i \geq r_i$, which also holds when $r_i = 0$. Quotient no. $r_i + 1$ is

$$q_i / (2r_i+1) = (r_i+s_i) / (2r_i+1)$$

This is greater than, equal to, or less than ½ exactly when the same holds for $s_i$. Quotients nos. $r_i + 2$, $r_i + 3$, and so on, are less than ½. The following conclusions can be drawn:

– If $s_i < ½$, then $k_i = k_i' = r_i$.

– If $s_i = ½$, then $k_i = r_i$ and $k_i' = r_i + 1$.

– If $s_i > ½$, then $k_i = k_i' = r_i + 1$.

This covers all cases.

Let $T$ be the number of entities $i$ for which $s_i > ½$, and let $T'$ be the number of entities $i$ for which $s_i \geq ½$. Then $K = R + T$, and $K' = R + T'$.

The number of entities $i$ for which $s_i > d^+$, is at most $S – 1$. At least one $i$ has $s_i = d^+$, and the number of entities $i$ for which $s_i \geq d^+$, is at least $S$.[43] The number of entities $i$ for which $s_i > d^–$, is at most $S$. At least one $i$ has $s_i = d^–$, and the number of entities $i$ for which $s_i \geq d^–$, is at least $S + 1$.

Note, moreover, that $c^+$ is quotient no. $H$ and $c^–$ is quotient no. $H + 1$.

A series of statements will now be proved concerning the relationship between $c^+$ and $d^+$, and concerning the relationship between $c^–$ and $d^–$.

(i) Assume $d^+ > ½$. If $s_i \geq d^+$, then $s_i > ½$. Hence $S \leq T$ and $H \leq K$. At least $H$ quotients are strictly greater than ½, and $c^+ > ½$.

(ii) Assume $d^+ \geq ½$. If $s_i \geq d^+$, then $s_i \geq ½$. Hence $S \leq T'$ and $H \leq K'$. At least $H$ quotients are greater than or equal to ½, and $c^+ \geq ½$.

(iii) Assume $d^– > ½$. If $s_i \geq d^–$, then $s_i > ½$. Hence $S + 1 \leq T$ and $H + 1 \leq K$. At least $H + 1$ quotients are strictly greater than ½, and $c^– > ½$.

(iv) Assume $d^– \geq ½$. If $s_i \geq d^–$, then $s_i \geq ½$. Hence $S + 1 \leq T'$ and $H + 1 \leq K'$. At least $H + 1$ quotients are greater than or equal to ½, and $c^– \geq ½$.

(v) Assume $d^+ < ½$. If $s_i \geq ½$, then $s_i > d^+$. Hence $T' \leq S – 1$ and $K' \leq H – 1$. At most $H – 1$ quotients are greater than or equal to ½, and $c^+ < ½$.

(vi) Assume $d^+ \leq ½$. If $s_i > ½$, then $s_i > d^+$. Hence $T \leq S – 1$ and $K \leq H – 1$. At most $H – 1$ quotients are strictly greater than ½, and $c^+ \leq ½$.

(vii) Assume $d^– < ½$. If $s_i \geq ½$, then $s_i > d^–$. Hence $T' \leq S$ and $K' \leq H$. At most $H$ quotients are greater than or equal to ½, and $c^– < ½$.

(viii) Assume $d^– \leq ½$. If $s_i > ½$, then $s_i > d^–$. Hence $T \leq S$ and $K \leq H$. At most $H$ quotients are strictly greater than ½, and $c^– \leq ½$.

It follows from (i), (ii), (v), and (vi) that $d^+$ is greater than, equal to, or less than ½ exactly when the same holds for $c^+$. Similarly, (iii), (iv), (vii), and (viii) imply that $d^–$ is greater than, equal to, or less than ½ exactly when the same holds for $c^–$. Note that it is not claimed that $d^+$ and $c^+$ are equal or that $d^–$ and $c^–$ are equal. Nor is it necessarily the case that the entity $i$ for which $s_i = d^+$ ($s_i = d^–$) is also the entity that won seat no. $H$ (seat no. $H + 1$) and therefore has

43. These statements do not necessarily hold when all the numbers $r_i$ are integers, in which case $S = 0$ and $d^+ = 1$. Then both methods give the perfectly proportional apportionment $h_i = r_i$. Moreover, $c^– < ½ < c^+$, and all the statements made below are true.

one of its quotients equal to $c^+$ ($c^-$).

Since $c^+ \geq c^-$ and $d^+ \geq d^-$, the following three cases are exhaustive and mutually exclusive, that is, exactly one of them must occur:

(a) $c^- > ½$ and $d^- > ½$

(b) $c^- \leq ½ \leq c^+$ and $d^- \leq ½ \leq d^+$

(c) $c^+ < ½$ and $d^+ < ½$

Case (b) can be further divided into two possibilities. They are mutually exclusive, and they are exhaustive within (b):

(b.1) $c^- \leq ½ \leq c^+$ and $c^- < c^+$, $d^- \leq ½ \leq d^+$ and $d^- < d^+$

(b.2) $c^- = c^+ = d^- = d^+ = ½$

Consider first case (b), in which $d^- \leq ½ \leq d^+$. If $h_i^L = r_i$, then $s_i \leq d^- \leq ½$, and if $h_i^L = r_i + 1$, then $s_i \geq d^+ \geq ½$. This gives $h_i^L \in [r_i+s_i]_{½} = [q_i]_{½}$. It follows from the discussion of (A3) that the $h_i^L$-vector is also an apportionment according to the odd numbers method. The number δ of (A3) can be chosen equal to 1. In (b.1), there is no tie for any of the methods, and the unique apportionment for the odd numbers method, given by the $h_i^O$-vector, is equal to the unique apportionment for the method of the largest remainder, given by the $h_i^L$-vector. In (b.2), there is a tie for both methods. The set of possible apportionments for the odd numbers method is equal to the set of possible apportionment for the method of the largest remainder.

Then consider case (a). If the apportionments given by the two methods are not equal, there must exist $i$ and $i'$ such that $h_i^O > h_i^L$ and $h_{i'}^O < h_{i'}^L$. Because $c^- > ½$, the number δ of (A3) must satisfy $\delta < 1$. Since $h_i^O \in [\delta q_i]_{½}$, $h_i^O$ is equal to $\delta q_i$ rounded upwards or downwards, that is, $h_i^O$ is equal to either $\lfloor \delta q_i \rfloor$ or $\lfloor \delta q_i \rfloor + 1$. From the description of the method of the largest remainder, it is clear that $h_i^L$ is equal to either $\lfloor q_i \rfloor$ or $\lfloor q_i \rfloor + 1$. Because $h_i^O > h_i^L$ and $\delta q_i < q_i$, this is only possible if $\lfloor \delta q_i \rfloor = \lfloor q_i \rfloor$, $h_i^O = \lfloor \delta q_i \rfloor + 1$ and $h_i^L = \lfloor q_i \rfloor$. Since $\lfloor \delta q_i \rfloor + 1 = h_i^O \in [\delta q_i]_{½}$, $\delta q_i \geq \lfloor \delta q_i \rfloor + ½$, while $h_i^L = \lfloor q_i \rfloor$ implies $q_i \leq \lfloor q_i \rfloor + d^- \leq \lfloor \delta q_i \rfloor + d^+$. These two inequalities give $q_i - \delta q_i \leq d^+ - ½$. Concerning entity $i'$, if $h_{i'}^O \geq \lfloor q_{i'} \rfloor$, then $h_{i'}^O < h_{i'}^L$ implies $h_{i'}^O = \lfloor q_{i'} \rfloor$ and $h_{i'}^L = \lfloor q_{i'} \rfloor + 1$. Therefore, $\delta q_{i'} \leq \lfloor q_{i'} \rfloor, + ½$ and $q_{i'} \geq \lfloor q_{i'} \rfloor, + d^+$, which gives $q_{i'} - \delta q_{i'} \geq d^+ - ½$. Otherwise, that is, if $h_{i'}^O < \lfloor q_{i'} \rfloor$, then $\delta q_{i'} \leq \lfloor q_{i'} \rfloor - ½$, which gives $q_{i'} - \delta q_{i'} \geq ½ \geq d^+ - ½$. In any case, $q_i - \delta q_i \leq d^+ - ½ \leq q_{i'} - \delta q_{i'}$. Since $\delta > 1$, this gives $q_i \leq q_{i'}$.

The conclusion is that if the two methods give different results, an entity favored by the odd numbers method cannot be bigger, measured by $p_i$ or $q_i$, than an entity favored by the method of the largest remainder. If there is a tie in one of the method (or both), the argument holds for any $h_i^O$-vector and any $h_i^L$-vector that are possible apportionment for the two methods.[44] That is, if it is possible that the odd numbers method favors entity $i$ while the method of the largest remainder favors entity $i'$, then $q_i \leq q_{i'}$.

In case (c), a similar argument shows that $h_i^O > h_i^L$ and $h_{i'}^O < h_{i'}^L$ imply $q_i \geq q_{i'}$.

To sum up, in case (b) the two methods give the same result. They can also give the same result in case (a), but if they do not, the odd numbers method is most favorable to small entities. In case (c), any difference must go in the direction of the odd numbers method being most favorable to large entities.

If $I = 2$, $d^- \leq ½ \leq d^+$ must necessarily hold. Hence case (b) applies. That is, the two

44. In this case, as opposed to case (b), there can be a tie for one of the methods and not for the other one.

methods are always equal when the seats are to be distributed between only *two* entities.

To illustrate the conclusions above, consider the example used to demonstrate the possibility of the Alabama paradox. For $H = 3$, $c^+ = 0.45 < ½$ and $d^+ = 0.36 < ½$, and case (c) applies. When the odd numbers method is used instead of the method of the largest remainder, one seat is moved from entity 3 to entity 1, that is, from the smallest to the largest entity. For $H = 4$, $c^+ \approx 0.57 > ½ > c^- = 0.48$ and $d^+ = 0.72 > ½ > d^- = 0.48$. Case (b) applies and the methods give the same apportionment.

## Table 1
**Population ($p_{i,j}$) by Cantons, Constituent Peoples, and groups**

| Canton | Bosniacs | Croats | Others | $p_{i,N}$ |
|---|---|---|---|---|
| **1** | 247,856 | 10,886 | 85,785 | 344,527 |
| **2** | 8,107 | 44,657 | 10,861 | 63,625 |
| **3** | 361,566 | 46,908 | 115,927 | 524,401 |
| **4** | 280,944 | 88,273 | 110,187 | 479,404 |
| **5** | 28,794 | 81 | 12,553 | 41,428 |
| **6** | 147,607 | 130,663 | 60,705 | 338,975 |
| **7** | 100,040 | 108,867 | 58,726 | 267,633 |
| **8** | 1,611 | 86,164 | 1,217 | 88,992 |
| **9** | 250,928 | 34,577 | 207,937 | 493,442 |
| **10** | 12,041 | 59,553 | 44,395 | 115,989 |
| $p_{M,j}$ | 1,439,494 | 610,629 | 708,293 | 2,758,416 |

## Table 2
**Normalized population ($q_{i,j}$) by Cantons, Constituent Peoples, and groups**

| Canton | Bosniacs | Croats | Others | $q_{i,N}$ | $r_i$ | $r_i^*$ |
|---|---|---|---|---|---|---|
| **1** | 7.1884 | 0.3157 | 2.4879 | 9.9920 | 9 | 10 |
| **2** | 0.2351 | 1.2951 | 0.3150 | 1.8452 | 3 | 3 |
| **3** | 10.4862 | 1.3604 | 3.3621 | 15.2087 | 14 | 15 |
| **4** | 8.1480 | 2.5601 | 3.1957 | 13.9038 | 13 | 13 |
| **5** | 0.8351 | 0.0023 | 0.3641 | 1.2015 | 3 | 3 |
| **6** | 4.2809 | 3.7895 | 1.7606 | 9.8310 | 10 | 9 |
| **7** | 2.9014 | 3.1574 | 1.7032 | 7.7620 | 9 | 7 |
| **8** | 0.0467 | 2.4989 | 0.0353 | 2.5809 | 3 | 3 |
| **9** | 7.2775 | 1.0028 | 6.0306 | 14.3109 | 13 | 14 |
| **10** | 0.3492 | 1.7272 | 1.2876 | 3.3640 | 3 | 3 |
| $q_{M,j}$ | 41.7485 | 17.7094 | 20.5421 | 80.0000 | 80 | 80 |
| $s_j$ | 30 | 30 | 20 | 80 | | |
| $s_j^*$ | 42 | 18 | 20 | 80 | | |

# Table 3.1: Bosniacs

**Population and quotients**

| Group | Population | Quotient I | | Quotient II | | Quotient III | | Quotient IV | | Quotient V | | Quotient VI | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **B 1** | 247,856 | 247,856 | 4 | 82,619 | 18 | 49,571 | 27 | 35,408 | 40 | 27,540 | 53 | 22,532 | (59) |
| **B 2** | 8,107 | 8,107 | | | | | | | | | | | |
| **B 3** | 361,566 | 361,566 | 1 | 120,522 | 8 | 72,313 | 19 | 51,652 | 25 | 40,174 | 34 | 32,870 | 43 |
| **B 4** | 280,944 | 280,944 | 2 | 93,648 | 13 | 56,189 | 24 | 40,135 | 35 | 31,216 | 44 | 25,540 | 55 |
| **B 5** | 28,794 | 28,794 | 48 | 9,598 | | | | | | | | | |
| **B 6** | 147,607 | 147,607 | 6 | 49,202 | 28 | 29,521 | 46 | 21,087 | (61) | 16,401 | | | |
| **B 7** | 100,040 | 100,040 | 12 | 33,347 | 42 | 20,008 | (62) | 14,291 | | | | | |
| **B 8** | 1,611 | 1,611 | | | | | | | | | | | |
| **B 9** | 250,928 | 250,928 | 3 | 83,643 | 17 | 50,186 | 26 | 35,847 | 39 | 27,881 | 51 | 22,812 | (59) |
| **B 10** | 12,041 | 12,041 | | | | | | | | | | | |
| **B Sum** | 1,439,494 | | | | | | | | | | | | |

| Group | Population | Quotient VII | | Quotient VIII | | Quotient IX | | Quotient X | | Quotient XI | | Quotient XII | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **B 1** | 247,856 | 19,066 | (64) | 16,524 | | | | | | | | | |
| **B 3** | 361,566 | 27,813 | 52 | 24,104 | 56 | 21,269 | (61) | 19,030 | (64) | 17,217 | (68) | 15,720 | |
| **B 4** | 280,944 | 21,611 | (61) | 18,730 | (65) | 16,526 | | | | | | | |
| **B 9** | 250,928 | 19,302 | (64) | 16,729 | | | | | | | | | |

## Table 3.2: Croats

### Population and quotients

| Group | Population | Quotient I | | Quotient II | | Quotient III | | Quotient IV | | Quotient V | | Quotient VI | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **C 1** | 10,886 | 10,886 | | | | | | | | | | | |
| **C 2** | 44,657 | 44,657 | 30 | 14,886 | | | | | | | | | |
| **C 3** | 46,908 | 46,908 | 29 | 15,636 | | | | | | | | | |
| **C 4** | 88,273 | 88,273 | 14 | 29,424 | 47 | 17,655 | 66 | 12,610 | | | | | |
| **C 5** | 81 | 81 | | | | | | | | | | | |
| **C 6** | 130,663 | 130,663 | 7 | 43,554 | 32 | 26,133 | 54 | 18,666 | 65 | 14,518 | | | |
| **C 7** | 108,867 | 108,867 | 11 | 36,289 | 38 | 21,773 | 60 | 15,552 | | | | | |
| **C 8** | 86,164 | 86,164 | 15 | 28,721 | 49 | 17,233 | 67 | 12,309 | | | | | |
| **C 9** | 34,577 | 34,577 | 41 | 11,526 | | | | | | | | | |
| **C 10** | 59,553 | 59,553 | 22 | 19,851 | 62 | 11,911 | | | | | | | |
| **C Sum** | 610,629 | | | | | | | | | | | | |

## Table 3.3: Others

### Population and quotients

| Group | Population | Quotient I | | Quotient II | | Quotient III | | Quotient IV | | Quotient V | | Quotient VI | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **O 1** | 85,785 | 85,785 | 16 | 28,595 | 50 | 17,157 | 68 | 12,255 | | | | | |
| **O 2** | 10,861 | 10,861 | | | | | | | | | | | |
| **O 3** | 115,927 | 115,927 | 9 | 38,642 | 36 | 23,185 | 57 | 16,561 | | | | | |
| **O 4** | 110,187 | 110,187 | 10 | 36,729 | 37 | 22,037 | 59 | 15,741 | | | | | |
| **O 5** | 12,553 | 12,553 | | | | | | | | | | | |
| **O 6** | 60,705 | 60,705 | 21 | 20,235 | 61 | 12,141 | | | | | | | |
| **O 7** | 58,726 | 58,726 | 23 | 19,575 | 63 | 11,745 | | | | | | | |
| **O 8** | 1,217 | 1,217 | | | | | | | | | | | |
| **O 9** | 207,937 | 207,937 | 5 | 69,312 | 20 | 41,587 | 33 | 29,705 | 45 | 23,104 | 58 | 18,903 | 64 |
| **O 10** | 44 395 | 44,395 | 31 | 14,798 | | | | | | | | | |
| **O Sum** | 708,293 | | | | | | | | | | | | |

| Group | Population | Quotient VII | |
|---|---|---|---|
| **O 9** | 207,937 | 15,995 | |

# Table 4
## Allocation of seats one by one based on population

| | Group | Quotient | G | P | C | Comment |
|---|---|---|---|---|---|---|
| **1** | **B 3** | 361,566 | 1 | 1 | 1 | |
| **2** | **B 4** | 280,944 | 1 | 2 | 1 | |
| **3** | **B 9** | 250,928 | 1 | 3 | 1 | |
| **4** | **B 1** | 247,856 | 1 | 4 | 1 | |
| **5** | **O 9** | 207,937 | 1 | 1 | 2 | |
| **6** | **B 6** | 147,607 | 1 | 5 | 1 | |
| **7** | **C 6** | 130,663 | 1 | 1 | 2 | |
| **8** | **B 3** | 120,522 | 2 | 6 | 2 | |
| **9** | **O 3** | 115,927 | 1 | 2 | 3 | |
| **10** | **O 4** | 110,187 | 1 | 3 | 2 | |
| **11** | **C 7** | 108,867 | 1 | 2 | 1 | |
| **12** | **B 7** | 100,040 | 1 | 7 | 2 | |
| **13** | **B 4** | 93,648 | 2 | 8 | 2 | |
| **14** | **C 4** | 88,273 | 1 | 3 | 3 | |
| **15** | **C 8** | 86,164 | 1 | 4 | 1 | |
| **16** | **O 1** | 85,785 | 1 | 4 | 2 | |
| **17** | **B 9** | 83,643 | 2 | 9 | 3 | |
| **18** | **B 1** | 82,619 | 2 | 10 | 3 | |
| **19** | **B 3** | 72,313 | 3 | 11 | 4 | |
| **20** | **O 9** | 69,312 | 2 | 5 | 4 | |
| **21** | **O 6** | 60,705 | 1 | 6 | 3 | |
| **22** | **C 10** | 59,553 | 1 | 5 | 1 | |
| **23** | **O 7** | 58,726 | 1 | 7 | 3 | |
| **24** | **B 4** | 56,189 | 3 | 12 | 4 | |
| **25** | **B 3** | 51,652 | 4 | 13 | 5 | |
| **26** | **B 9** | 50,186 | 3 | 14 | 5 | |
| **27** | **B 1** | 49,571 | 3 | 15 | 4 | |
| **28** | **B 6** | 49,202 | 2 | 16 | 4 | |
| **29** | **C 3** | 46,908 | 1 | 6 | 6 | |
| **30** | **C 2** | 44,657 | 1 | 7 | 1 | |
| **31** | **O 10** | 44,395 | 1 | 8 | 2 | |
| **32** | **C 6** | 43,554 | 2 | 8 | 5 | |
| **33** | **O 9** | 41,587 | 3 | 9 | 6 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| **34** | **B 3** | 40,174 | 5 | 17 | 7 | |
| **35** | **B 4** | 40,135 | 4 | 18 | 5 | |
| **36** | **O 3** | 38,642 | 2 | 10 | 8 | |
| **37** | **O 4** | 36,729 | 2 | 11 | 6 | |
| **38** | **C 7** | 36,289 | 2 | 9 | 4 | |
| **39** | **B 9** | 35,847 | 4 | 19 | 7 | |
| **40** | **B 1** | 35,408 | 4 | 20 | 5 | |
| **41** | **C 9** | 34,577 | 1 | 10 | 8 | |
| **42** | **B 7** | 33,347 | 2 | 21 | 5 | |
| **43** | **B 3** | 32,870 | 6 | 22 | 9 | |
| **44** | **B 4** | 31,216 | 5 | 23 | 7 | |
| **45** | **O 9** | 29,705 | 4 | 12 | 9 | |
| **46** | **B 6** | 29,521 | 3 | 24 | 6 | |
| **47** | **C 4** | 29,424 | 2 | 11 | 8 | |
| **48** | **B 5** | 28,794 | 1 | 25 | 1 | |
| **49** | **C 8** | 28,721 | 2 | 12 | 2 | |
| **50** | **O 1** | 28,595 | 2 | 13 | 6 | |
| **51** | **B 9** | 27,881 | 5 | 26 | 10 | |
| **52** | **B 3** | 27,813 | 7 | 27 | 10 | |
| **53** | **B 1** | 27,540 | 5 | 28 | 7 | |
| **54** | **C 6** | 26,133 | 3 | 13 | 7 | |
| **55** | **B 4** | 25,540 | 6 | 29 | 9 | |
| **56** | **B 3** | 24,104 | 8 | 30 | 11 | Bosniacs can get no more seats |
| **57** | **O 3** | 23,185 | 3 | 14 | 12 | |
| **58** | **O 9** | 23,104 | 5 | 15 | 11 | |
| | **B 9** | 22,812 | (6) | (31) | | Bosniacs ineligible |
| | **B 1** | 22,532 | (6) | (32) | | Bosniacs ineligible |
| **59** | **O 4** | 22,037 | 3 | 16 | 10 | |
| **60** | **C 7** | 21,773 | 3 | 14 | 6 | |
| | **B 4** | 21,611 | (7) | (33) | | Bosniacs ineligible |
| | **B 3** | 21,269 | (9) | (34) | | Bosniacs ineligible |
| | **B 6** | 21,087 | (4) | (35) | | Bosniacs ineligible |
| **61** | **O 6** | 20,235 | 2 | 17 | 8 | |
| | **B 7** | 20,008 | (3) | (36) | | Bosniacs ineligible |
| **62** | **C 10** | 19,851 | 2 | 18 | 3 | Canton 10 can get no more seats |
| **63** | **O 7** | 19,575 | 2 | 18 | 7 | |
| | **B 9** | 19,302 | (7) | (37) | | Bosniacs ineligible |

| | | | | | | |
|---|---|---|---|---|---|---|
| | **B 1** | 19,066 | (7) | (38) | | Bosniacs ineligible |
| | **B 3** | 19,030 | (10) | (39) | | Bosniacs ineligible |
| **64** | **O 9** | 18,903 | 6 | 19 | 12 | |
| | **B 4** | 18,730 | (8) | (40) | | Bosniacs ineligible |
| **65** | **C 6** | 18,666 | 4 | 19 | 9 | |
| **66** | **C 4** | 17,655 | 3 | 20 | 11 | |
| **67** | **C 8** | 17,233 | 3 | 21 | 3 | Canton 8 can get no more seats |
| | **B 3** | 17,217 | (11) | (41) | | Bosniacs ineligible |
| **68** | **O 1** | 17,157 | 3 | 20 | | Others can get no more seats |

## Table 5
**Allocation of the first 68 seats based on population**

| Canton | Bosniacs | Croats | Others | Total |
|---|---|---|---|---|
| **1** | 5 | | 3 | 8 |
| **2** | | 1 | | 1 |
| **3** | 8 | 1 | 3 | 12 |
| **4** | 6 | 3 | 3 | 12 |
| **5** | 1 | | | 1 |
| **6** | 3 | 4 | 2 | 9 |
| **7** | 2 | 3 | 2 | 7 |
| **8** | | 3 | | 3 |
| **9** | 5 | 1 | 6 | 12 |
| **10** | | 2 | 1 | 3 |
| **Total** | 30 | 18 | 20 | 68 |

## Table 6
**Final apportionment based on population**

| Canton | Bosniacs | Croats | Others | Total |
|---|---|---|---|---|
| **1** | 5 | 1 | 3 | 9 |
| **2** | | 3 | | 3 |
| **3** | 8 | 3 | 3 | 14 |
| **4** | 6 | 4 | 3 | 13 |
| **5** | 1 | 2 | | 3 |
| **6** | 3 | 5 | 2 | 10 |
| **7** | 2 | 5 | 2 | 9 |
| **8** | | 3 | | 3 |
| **9** | 5 | 2 | 6 | 13 |
| **10** | | 2 | 1 | 3 |
| **Total** | 30 | 30 | 20 | 80 |

## Table 7
**Population normalized by Canton ($q_{i,j}^{C}$)**

| Canton | Bosniacs | Croats | Others | $q_{i,N}^{C}$ | $r_i$ |
|---|---|---|---|---|---|
| **1** | 6.4747 | 0.2844 | 2.2409 | 9.0000 | 9 |
| **2** | 0.3823 | 2.1056 | 0.5121 | 3.0000 | 3 |
| **3** | 9.6528 | 1.2523 | 3.0949 | 14.0000 | 14 |
| **4** | 7.6184 | 2.3937 | 2.9879 | 13.0000 | 13 |
| **5** | 2.0851 | 0.0059 | 0.9090 | 3.0000 | 3 |
| **6** | 4.3545 | 3.8547 | 1.7908 | 10.0000 | 10 |
| **7** | 3.3642 | 3.6610 | 1.9748 | 9.0000 | 9 |
| **8** | 0.0543 | 2.9047 | 0.0410 | 3.0000 | 3 |
| **9** | 6.6108 | 0.9110 | 5.4782 | 13.0000 | 13 |
| **10** | 0.3114 | 1.5403 | 1.1483 | 3.0000 | 3 |
| $q_{M,j}^{C}$ | 40.9085 | 18.9136 | 20.1779 | 80.0000 | 80 |
| $s_j$ | 30 | 30 | 20 | 80 | |

## Table 8
**Final apportionment based on normalization by Canton**

| Canton | Bosniacs | Croats | Others | Total |
|---|---|---|---|---|
| **1** | 5 | 2 | 2 | 9 |
| **2** | | 2 | 1 | 3 |
| **3** | 7 | 4 | 3 | 14 |
| **4** | 6 | 4 | 3 | 13 |
| **5** | 2 | | 1 | 3 |
| **6** | 3 | 5 | 2 | 10 |
| **7** | 2 | 5 | 2 | 9 |
| **8** | | 3 | | 3 |
| **9** | 5 | 3 | 5 | 13 |
| **10** | | 2 | 1 | 3 |
| **Total** | 30 | 30 | 20 | 80 |

## Table 9
**Population normalized by Constituent People ($q_{i,j}^P$)**

| Canton | Bosniacs | Croats | Others | $q_{i,N}^P$ | $r_i$ |
|---|---|---|---|---|---|
| **1** | 5.1655 | 0.5348 | 2.4223 | 8.1226 | 9 |
| **2** | 0.1690 | 2.1940 | 0.3067 | 2.6697 | 3 |
| **3** | 7.5353 | 2.3046 | 3.2734 | 13.1133 | 14 |
| **4** | 5.8551 | 4.3368 | 3.1113 | 13.3032 | 13 |
| **5** | 0.6001 | 0.0040 | 0.3545 | 0.9586 | 3 |
| **6** | 3.0762 | 6.4194 | 1.7141 | 11.2097 | 10 |
| **7** | 2.0849 | 5.3486 | 1.6582 | 9.0917 | 9 |
| **8** | 0.0336 | 4.2332 | 0.0344 | 4.3012 | 3 |
| **9** | 5.2295 | 1.6988 | 5.8715 | 12.7998 | 13 |
| **10** | 0.2509 | 2.9258 | 1.2536 | 4.4303 | 3 |
| $q_{M,j}^P$ | 30.0001 | 30.0000 | 20.0000 | 80.0001 | 80 |
| $s_j$ | 30 | 30 | 20 | 80 | |

## Table 10
**Final apportionment based on normalization by Constituent People**

| Canton | Bosniacs | Croats | Others | Total |
|---|---|---|---|---|
| **1** | 5 | 1 | 3 | 9 |
| **2** | | 3 | | 3 |
| **3** | 8 | 2 | 4 | 14 |
| **4** | 6 | 4 | 3 | 13 |
| **5** | 1 | 2 | | 3 |
| **6** | 3 | 6 | 1 | 10 |
| **7** | 2 | 5 | 2 | 9 |
| **8** | | 3 | | 3 |
| **9** | 5 | 2 | 6 | 13 |
| **10** | | 2 | 1 | 3 |
| **Total** | 30 | 30 | 20 | 80 |

## Table 11
**Continuous allocation ($a_{i,j}$)**

| **Canton** | **Bosniacs** | **Croats** | **Others** | $a_{i,N}$ | $\lambda_i^c$ |
|---|---|---|---|---|---|
| **1** | 5.5831 | 0.7131 | 2.7038 | 9.0000 | 1.1877 |
| **2** | 0.1588 | 2.5436 | 0.2977 | 3.0001 | 1.0327 |
| **3** | 7.6674 | 2.8927 | 3.4399 | 14.0000 | 1.1182 |
| **4** | 5.2792 | 4.8237 | 2.8972 | 13.0001 | 0.9908 |
| **5** | 1.8539 | 0.0152 | 1.1309 | 3.0000 | 3.3949 |
| **6** | 2.4098 | 6.2034 | 1.3867 | 9.9999 | 0.8608 |
| **7** | 1.8050 | 5.7123 | 1.4827 | 9.0000 | 0.9514 |
| **8** | 0.0190 | 2.9608 | 0.0201 | 2.9999 | 0.6231 |
| **9** | 5.0777 | 2.0347 | 5.8876 | 13.0000 | 1.0670 |
| **10** | 0.1460 | 2.1005 | 0.7534 | 2.9999 | 0.6395 |
| $a_{M,j}$ | 29.9999 | 30.0000 | 20.0000 | 79.9999 | |
| $\mu_j^c$ | 0.6539 | 1.9016 | 0.9150 | | |

## Table 12.1: Bosniacs

### Continuous allocation and quotients

| Group | Allocation | Quotient I | | Quotient II | | Quotient III | | Quotient IV | | Quotient V | | Quotient VI | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **B 1** | 5.5831 | 5.5831 | 5 | 1.8610 | 22 | 1.1166 | 36 | 0.7976 | 51 | 0.6203 | 64 | 0.5076 | (79) |
| **B 2** | 0.1588 | 0.1588 | | | | | | | | | | | |
| **B 3** | 7.6674 | 7.6674 | 1 | 2.5558 | 14 | 1.5335 | 28 | 1.0953 | 37 | 0.8519 | 46 | 0.6970 | 57 |
| **B 4** | 5.2792 | 5.2792 | 6 | 1.7597 | 25 | 1.0558 | 38 | 0.7542 | 52 | 0.5866 | 69 | 0.4799 | |
| **B 5** | 1.8539 | 1.8539 | 23 | 0.6180 | 65 | 0.3708 | | | | | | | |
| **B 6** | 2.4098 | 2.4098 | 16 | 0.8033 | 50 | 0.4820 | 79 | 0.3443 | | | | | |
| **B 7** | 1.8050 | 1.8050 | 24 | 0.6017 | 66 | 0.3610 | | | | | | | |
| **B 8** | 0.0190 | 0.0190 | | | | | | | | | | | |
| **B 9** | 5.0777 | 5.0777 | 7 | 1.6926 | 26 | 1.0155 | 39 | 0.7254 | 54 | 0.5642 | 72 | 0.4616 | |
| **B 10** | 0.1460 | 0.1460 | | | | | | | | | | | |
| **B Sum** | 29.9999 | | | | | | | | | | | | |

| Group | Allocation | Quotient VII | | Quotient VIII | | Quotient IX | |
|---|---|---|---|---|---|---|---|
| **B 1** | 5.5831 | 0.4295 | | | | | |
| **B 3** | 7.6674 | 0.5898 | 68 | 0.5112 | 78 | 0.4510 | |

# Table 12.2: Croats

## Continuous allocation and quotients

| Group | Allocation | Quotient I | | Quotient II | | Quotient III | | Quotient IV | | Quotient V | | Quotient VI | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **C 1** | 0.7131 | 0.7131 | 55 | 0.2377 | | | | | | | | | |
| **C 2** | 2.5436 | 2.5436 | 15 | 0.8479 | 47 | 0.5087 | (79) | 0.3634 | | | | | |
| **C 3** | 2.8927 | 2.8927 | 12 | 0.9642 | 43 | 0.5785 | 71 | 0.4132 | | | | | |
| **C 4** | 4.8237 | 4.8237 | 8 | 1.6079 | 27 | 0.9647 | 42 | 0.6891 | 59 | 0.5360 | 75 | 0.0648 | |
| **C 5** | 0.0152 | 0.0152 | | | | | | | | | | | |
| **C 6** | 6.2034 | 6.2034 | 2 | 2.0678 | 18 | 1.2407 | 31 | 0.8862 | 45 | 0.6893 | 58 | 0.5639 | 73 |
| **C 7** | 5.7123 | 5.7123 | 4 | 1.9041 | 21 | 1.1425 | 34 | 0.8160 | 49 | 0.6347 | 63 | 0.5193 | 77 |
| **C 8** | 2.9608 | 2.9608 | 10 | 0.9869 | 40 | 0.5922 | 67 | 0.4230 | | | | | |
| **C 9** | 2.0347 | 2.0347 | 19 | 0.6782 | 61 | 0.4069 | | | | | | | |
| **C 10** | 2.1005 | 2.1005 | 17 | 0.7002 | 56 | 0.4201 | | | | | | | |
| **C Sum** | 30.0000 | | | | | | | | | | | | |

| Group | Allocation | Quotient VII | |
|---|---|---|---|
| **C 6** | 6.2034 | 0.4772 | |
| **C 7** | 5.7123 | 0.4394 | |

## Table 12.3: Others

### Continuous allocation and quotients

| Group | Allocation | Quotient I | | Quotient II | | Quotient III | | Quotient IV | | Quotient V | | Quotient VI | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **O 1** | 2.7038 | 2.7038 | 13 | 0.9013 | 44 | 0.5408 | 74 | 0.3863 | | | | | |
| **O 2** | 0.2977 | 0.2977 | | | | | | | | | | | |
| **O 3** | 3.4399 | 3.4399 | 9 | 1.1466 | 33 | 0.6880 | 60 | 0.4914 | (79) | 0.3822 | | | |
| **O 4** | 2.8972 | 2.8972 | 11 | 0.9657 | 41 | 0.5794 | 70 | 0.4139 | | | | | |
| **O 5** | 1.1309 | 1.1309 | 35 | 0.3770 | | | | | | | | | |
| **O 6** | 1.3867 | 1.3867 | 30 | 0.4622 | | | | | | | | | |
| **O 7** | 1.4827 | 1.4827 | 29 | 0.4942 | (79) | 0.2965 | | | | | | | |
| **O 8** | 0.0201 | 0.0201 | | | | | | | | | | | |
| **O 9** | 5.8876 | 5.8876 | 3 | 1.9625 | 20 | 1.1775 | 32 | 0.8411 | 48 | 0.6542 | 62 | 0.5352 | 76 |
| **O 10** | 0.7534 | 0.7534 | 53 | 0.2511 | | | | | | | | | |
| **O Sum** | 20.0000 | | | | | | | | | | | | |

| Group | Allocation | Quotient VII | |
|---|---|---|---|
| **O 9** | 5.8876 | 0.4529 | |

# Table 13

## Allocation of seats one by one based on continuous allocation

| | Group | Quotient | G | P | C | Comment |
|---|---|---|---|---|---|---|
| **1** | **B 3** | 7.6674 | 1 | 1 | 1 | |
| **2** | **C 6** | 6.2034 | 1 | 1 | 1 | |
| **3** | **O 9** | 5.8876 | 1 | 1 | 1 | |
| **4** | **C 7** | 5.7123 | 1 | 2 | 1 | |
| **5** | **B 1** | 5.5831 | 1 | 2 | 1 | |
| **6** | **B 4** | 5.2792 | 1 | 3 | 1 | |
| **7** | **B 9** | 5.0777 | 1 | 4 | 2 | |
| **8** | **C 4** | 4.8237 | 1 | 3 | 2 | |
| **9** | **O 3** | 3.4399 | 1 | 2 | 2 | |
| **10** | **C 8** | 2.9608 | 1 | 4 | 1 | |
| **11** | **O 4** | 2.8972 | 1 | 3 | 3 | |
| **12** | **C 3** | 2.8927 | 1 | 5 | 3 | |
| **13** | **O 1** | 2.7038 | 1 | 4 | 2 | |
| **14** | **B 3** | 2.5558 | 2 | 5 | 4 | |
| **15** | **C 2** | 2.5436 | 1 | 6 | 1 | |
| **16** | **B 6** | 2.4098 | 1 | 6 | 2 | |
| **17** | **C 10** | 2.1005 | 1 | 7 | 1 | |
| **18** | **C 6** | 2.0678 | 2 | 8 | 3 | |
| **19** | **C 9** | 2.0347 | 1 | 9 | 3 | |
| **20** | **O 9** | 1.9625 | 2 | 5 | 4 | |
| **21** | **C 7** | 1.9041 | 2 | 10 | 2 | |
| **22** | **B 1** | 1.8610 | 2 | 7 | 3 | |
| **23** | **B 5** | 1.8539 | 1 | 8 | 1 | |
| **24** | **B 7** | 1.8050 | 1 | 9 | 3 | |
| **25** | **B 4** | 1.7597 | 2 | 10 | 4 | |
| **26** | **B 9** | 1.6926 | 2 | 11 | 5 | |
| **27** | **C 4** | 1.6079 | 2 | 11 | 5 | |
| **28** | **B 3** | 1.5335 | 3 | 12 | 5 | |
| **29** | **O 7** | 1.4827 | 1 | 6 | 4 | |
| **30** | **O 6** | 1.3867 | 1 | 7 | 4 | |
| **31** | **C 6** | 1.2407 | 3 | 12 | 5 | |
| **32** | **O 9** | 1.1775 | 3 | 8 | 6 | |
| **33** | **O 3** | 1.1466 | 2 | 9 | 6 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| **34** | **C 7** | 1.1425 | 3 | 13 | 5 | |
| **35** | **O 5** | 1.1309 | 1 | 10 | 2 | |
| **36** | **B 1** | 1.1166 | 3 | 13 | 4 | |
| **37** | **B 3** | 1.0953 | 4 | 14 | 7 | |
| **38** | **B 4** | 1.0558 | 3 | 15 | 6 | |
| **39** | **B 9** | 1.0155 | 3 | 16 | 7 | |
| **40** | **C 8** | 0.9869 | 2 | 14 | 2 | |
| **41** | **O 4** | 0.9657 | 2 | 11 | 7 | |
| **42** | **C 4** | 0.9647 | 3 | 15 | 8 | |
| **43** | **C 3** | 0.9642 | 2 | 16 | 8 | |
| **44** | **O 1** | 0.9013 | 2 | 12 | 5 | |
| **45** | **C 6** | 0.8862 | 4 | 17 | 6 | |
| **46** | **B 3** | 0.8519 | 5 | 17 | 9 | |
| **47** | **C 2** | 0.8479 | 2 | 18 | 2 | |
| **48** | **O 9** | 0.8411 | 4 | 13 | 8 | |
| **49** | **C 7** | 0.8160 | 4 | 19 | 6 | |
| **50** | **B 6** | 0.8033 | 2 | 18 | 7 | |
| **51** | **B 1** | 0.7976 | 4 | 19 | 6 | |
| **52** | **B 4** | 0.7542 | 4 | 20 | 9 | |
| **53** | **O 10** | 0.7534 | 1 | 14 | 2 | |
| **54** | **B 9** | 0.7254 | 4 | 21 | 9 | |
| **55** | **C 1** | 0.7131 | 1 | 20 | 7 | |
| **56** | **C 10** | 0.7002 | 2 | 21 | 3 | Canton 10 can get no more seats |
| **57** | **B 3** | 0.6970 | 6 | 22 | 10 | |
| **58** | **C 6** | 0.6893 | 5 | 22 | 8 | |
| **59** | **C 4** | 0.6891 | 4 | 23 | 10 | |
| **60** | **O 3** | 0.6880 | 3 | 15 | 11 | |
| **61** | **C 9** | 0.6782 | 2 | 24 | 10 | |
| **62** | **O 9** | 0.6542 | 5 | 16 | 11 | |
| **63** | **C 7** | 0.6347 | 5 | 25 | 7 | |
| **64** | **B 1** | 0.6203 | 5 | 23 | 8 | |
| **65** | **B 5** | 0.6180 | 2 | 24 | 3 | Canton 5 can get no more seats |
| **66** | **B 7** | 0.6017 | 2 | 25 | 8 | |
| **67** | **C 8** | 0.5922 | 3 | 26 | 3 | Canton 8 can get no more seats |
| **68** | **B 3** | 0.5898 | 7 | 26 | 12 | |
| **69** | **B 4** | 0.5866 | 5 | 27 | 11 | |
| **70** | **O 4** | 0.5794 | 3 | 17 | 12 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| **71** | **C 3** | 0.5785 | 3 | 27 | 13 | |
| **72** | **B 9** | 0.5642 | 5 | 28 | 12 | |
| **73** | **C 6** | 0.5639 | 6 | 28 | 9 | |
| **74** | **O 1** | 0.5408 | 3 | 18 | 9 | Canton 1 can get no more seats |
| **75** | **C 4** | 0.5360 | 5 | 29 | 13 | Canton 4 can get no more seats |
| **76** | **O 9** | 0.5352 | 6 | 19 | 13 | Canton 9 can get no more seats |
| **77** | **C 7** | 0.5193 | 6 | 30 | 9 | Croats can get no more seats<br>Canton 7 can get no more seats |
| **78** | **B 3** | 0.5112 | 8 | 29 | 14 | Canton 3 can get no more seats |
| | **C 2** | 0.5087 | (3) | (31) | | Croats ineligible |
| | **B 1** | 0.5076 | (6) | | (10) | Canton 1 ineligible |
| | **O 7** | 0.4942 | (2) | | (10) | Canton 7 ineligible |
| | **O 3** | 0.4914 | (4) | | (15) | Canton 3 ineligible |
| **79** | **B 6** | 0.4820 | 3 | 30 | 10 | Bosniacs can get no more seats<br>Canton 6 can get no more seats |

## Table 14
**Distribution of the first 79 seats based on continuous allocation**

| Canton | Bosniacs | Croats | Others | Total |
|---|---|---|---|---|
| **1** | 5 | 1 | 3 | 9 |
| **2** | | 2 | | 2 |
| **3** | 8 | 3 | 3 | 14 |
| **4** | 5 | 5 | 3 | 13 |
| **5** | 2 | | 1 | 3 |
| **6** | 3 | 6 | 1 | 10 |
| **7** | 2 | 6 | 1 | 9 |
| **8** | | 3 | | 3 |
| **9** | 5 | 2 | 6 | 13 |
| **10** | | 2 | 1 | 3 |
| **Total** | 30 | 30 | 19 | 79 |

## Table 15
**Final apportionment based on continuous allocation**

| Canton | Bosniacs | Croats | Others | Total |
|---|---|---|---|---|
| **1** | 5 | 1 | 3 | 9 |
| **2** | | 2 | 1 | 3 |
| **3** | 8 | 3 | 3 | 14 |
| **4** | 5 | 5 | 3 | 13 |
| **5** | 2 | | 1 | 3 |
| **6** | 3 | 6 | 1 | 10 |
| **7** | 2 | 6 | 1 | 9 |
| **8** | | 3 | | 3 |
| **9** | 5 | 2 | 6 | 13 |
| **10** | | 2 | 1 | 3 |
| **Total** | 30 | 30 | 20 | 80 |

## Table 16

**Basis for direct solution of the discrete problem ($\lambda_i \cdot \mu_j \cdot q_{i,j}$)**

| Canton | Bosniacs | Croats | Others | $\lambda_i^d$ |
|---|---|---|---|---|
| **1** | 5.4999 | 0.6763 | 2.6949 | 1.1700 |
| **2** | 0.1621 | 2.5001 | 0.3074 | 1.0543 |
| **3** | 7.6674 | 2.7852 | 3.4803 | 1.1182 |
| **4** | 5.2792 | 4.6443 | 2.9313 | 0.9908 |
| **5** | 1.8539 | 0.0146 | 1.1442 | 3.3949 |
| **6** | 2.5001 | 6.1965 | 1.4556 | 0.8931 |
| **7** | 1.8050 | 5.4999 | 1.5001 | 0.9514 |
| **8** | 0.0190 | 2.8507 | 0.0204 | 0.6231 |
| **9** | 5.0777 | 1.9591 | 5.9569 | 1.0670 |
| **10** | 0.1460 | 2.0224 | 0.7623 | 0.6395 |
| $\mu_j^d$ | 0.6539 | 1.8309 | 0.9258 | |

## Table 17

**Final apportionment when the discrete problem is solved directly**

| Canton | Bosniacs | Croats | Others | Total |
|---|---|---|---|---|
| **1** | 5 | 1 | 3 | 9 |
| **2** | | 3 | | 3 |
| **3** | 8 | 3 | 3 | 14 |
| **4** | 5 | 5 | 3 | 13 |
| **5** | 2 | | 1 | 3 |
| **6** | 3 | 6 | 1 | 10 |
| **7** | 2 | 5 | 2 | 9 |
| **8** | | 3 | | 3 |
| **9** | 5 | 2 | 6 | 13 |
| **10** | | 2 | 1 | 3 |
| **Total** | 30 | 30 | 20 | 80 |

## Table 18
**Comparison of apportionment by five methods**

| **Canton** | **Bosniacs** | | | | | **Croats** | | | | | **Others** | | | | | $r_j$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **0** | **C** | **P** | **T** | **D** | **0** | **C** | **P** | **T** | **D** | **0** | **C** | **P** | **T** | **D** | |
| **1** | 5 | 5 | 5 | 5 | 5 | 1 | 2 | 1 | 1 | 1 | 3 | 2 | 3 | 3 | 3 | 9 |
| **2** | | | | | | 3 | 2 | 3 | 2 | 3 | | 1 | | 1 | | 3 |
| **3** | 8 | 7 | 8 | 8 | 8 | 3 | 4 | 2 | 3 | 3 | 3 | 3 | 4 | 3 | 3 | 14 |
| **4** | 6 | 6 | 6 | 5 | 5 | 4 | 4 | 4 | 5 | 5 | 3 | 3 | 3 | 3 | 3 | 13 |
| **5** | 1 | 2 | 1 | 2 | 2 | 2 | | 2 | | | | 1 | | 1 | 1 | 3 |
| **6** | 3 | 3 | 3 | 3 | 3 | 5 | 5 | 6 | 6 | 6 | 2 | 2 | 1 | 1 | 1 | 10 |
| **7** | 2 | 2 | 2 | 2 | 2 | 5 | 5 | 5 | 6 | 5 | 2 | 2 | 2 | 1 | 2 | 9 |
| **8** | | | | | | 3 | 3 | 3 | 3 | 3 | | | | | | 3 |
| **9** | 5 | 5 | 5 | 5 | 5 | 2 | 3 | 2 | 2 | 2 | 6 | 5 | 6 | 6 | 6 | 13 |
| **10** | | | | | | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 3 |
| **Total** | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 20 | 20 | 20 | 20 | 20 | 80 |

**0** Population, no normalization (Section 3, Table 6)
**C** Normalization by Canton (Section 4, Table 8)
**P** Normalization by Constituent People (Section 4, Table 10)
**T** Two-dimensional normalization (Sections 5 and 6, Table 15)
**D** Discrete solution (Sections 7 and 8, Table 17)

## Table 19
### Distance between apportionments

| | 0 | C | P | T |
|---|---|---|---|---|
| **C** | 6 | | | |
| **P** | 2 | 8 | | |
| **T** | 6 | 6 | 6 | |
| **D** | 4 | 6 | 4 | 2 |

**0** Population, no normalization (Section 3, Table 6)
**C** Normalization by Canton (Section 4, Table 8)
**P** Normalization by Constituent People (Section 4, Table 10)
**T** Two-dimensional normalization (Sections 5 and 6, Table 15)
**D** Discrete solution (Sections 7 and 8, Table 17)