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Multidimensional political apportionment

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Deciding how to allocate the seats of a deliberative assembly is one of the most fundamental problems in the political organization of societies and has been widely studied over two centuries already. The idea of proportionality is at the core of most approaches to tackle this problem, and this notion is captured by the divisor methods, such as the Jefferson/D'Hondt method. In a seminal work, Balinski and Demange extended the single-dimensional idea of divisor methods to the setting in which the seat allocation is simultaneously determined by two dimensions and proposed the so-called biproportional apportionment method. The method, currently used in several electoral systems, is, however, limited to two dimensions and the question of extending it is considered to be an important problem both theoretically and in practice. In this work we initiate the study of multidimensional proportional apportionment. We first formalize a notion of multidimensional proportionality that naturally extends that of Balinski and Demange. By means of analyzing an appropriate integer linear program we are able to prove that, in contrast to the two-dimensional case, the existence of multidimensional proportional apportionments is not guaranteed and deciding their existence is a computationally hard problem (NP-complete). Interestingly, our main result asserts that it is possible to find approximate multidimensional proportional apportionments that deviate from the marginals by a small amount. The proof arises through the lens of discrepancy theory, mainly inspired by the celebrated Beck–Fiala theorem. We finally evaluate our approach by using the data from the recent 2021 Chilean Constitutional Convention election.

apportionment | integer programming | social choice

A cornerstone of modern democracies is the division of the political organization, generally including a deliberative assembly with the goal of reflecting the needs of different segments across the population. In the apportionment problem, the purpose is to allocate the total number of seats in a deliberative assembly, and how to solve this problem is something that has been discussed and studied extensively in modern history. A natural goal that is at the core of many apportionment systems is the idea of proportionality. That is, a party receives an amount of seats that is proportional to the number of votes that the party obtained in the election. Since in general the seats are not divisible, it is necessary to properly formalize the notion of proportionality in an integral setting. The divisor methods provide an answer to this problem, based on appropriately scaling the votes and rounding the result to meet the house size. These methods are widely used at national and regional levels in many democracies around the world. The two most common versions are, by far, the Jefferson/D'Hondt method proposed by Thomas Jefferson in 1792 and the Webster/Sainte-Laguë method first proposed by Daniel Webster in 1832. While both methods correspond to divisor methods, the latter rounds to the nearest integer while the former takes integer part (1).

In their seminal work, Balinski and Demange (2, 3) extended the notion of proportionality and divisor methods to the case in which the apportionment is ruled by two dimensions, studying this extension from an axiomatic and algorithmic point of view. In this setting, an instance is given by an integral matrix (of votes) where the rows typically represent the political parties and the columns represent the regions. We are also given a list of strictly positive integers, called marginals, specifying the row and column sums for any feasible biproportional apportionment. Thus the row marginals account for the number of seats that have to be allocated to the corresponding party, and the column marginals correspond to the number of seats a given district should get.* The goal is to find a matrix (of seats) satisfying the marginals and keeping proportionality with respect to the votes simultaneously in both dimensions. This notion is captured by a set of multipliers,

*Generally, party marginals correspond to the proportion of seats each party should get, given its proportion on the number of votes, by applying some divisor method. Sometimes, however, a certain minimum proportion of the votes is imposed to obtain seats in the house. District marginals, on the other hand, are often proportional to the population of each district.

Significance

A cornerstone in the modern political organization of societies is the existence of a deliberative assembly, reflecting the needs of different population segments. As modern societies become more complex, representation according to dimensions beyond political affiliation and geography is demanded; examples include gender balance and ethnicity. As this dimensionality increases, the task becomes more challenging and requires more sophisticated mathematical tools. In this paper, we initiate the study of multidimensional apportionments and show that, in three and more dimensions, their existence is not guaranteed. However, our main result states that it is possible to elect a house nearly respecting proportionality of representation along several dimensions simultaneously. We finally illustrate the potential of our approach with recent election data.

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one for each row and column. The biproportional apportionment method is currently used in elections of several cantons in Switzerland.[†]

A distinctive feature of the biproportional method is that the existence (and essential uniqueness) is guaranteed (2), under very natural conditions. However, by design, the biproportional method is limited to the case of apportionments ruled by two dimensions. Departing from the two-dimensional case is not only a challenging mathematical question but also a relevant practical problem. Indeed, as modern societies become more complex, representation of dimensions beyond political affiliation and geography is increasingly demanded (4). For instance, New Zealand's parliament includes ethnic representation while the recently elected 2021 Chilean Constitutional Convention includes gender balance as a constraint. Another example, mentioned by Demange (5), is the proposed division of three types of "constituent people" (Bosniacs, Croats, and Others) in the Parliament of the Federation of Bosnia and Herzegovina, which led Demange to raise the multidimensional proportional apportionment as a challenging question.

In this paper, we initiate the study of multidimensional proportional apportionment and establish that if we allow small deviations from the prescribed marginals, then existence is again guaranteed. As an illustration, in the case of three dimensions, say political, regional, and gender, our main theorem states that there exists a three-dimensional proportional apportionment that deviates by at most one from each of the prescribed marginals. We remark that using such a method in practice requires defining precisely how the marginals are computed for each dimension and designing the ballots appropriately. For instance, an open-list ballot could be used where the party and gender of each candidate are specified, while the marginals may come from legal mandates. Alternatively, among other options, a closed-list ballot with parties' lists separated by gender could be used.

Our Contribution. One of the key technical ingredients used to study the biproportional apportionment corresponds to a linear program, inspired by the closely related matrix-scaling problem (6–8). Following this approach, we introduce an integer linear program to analyze the multidimensional setting and provide structural results by studying its linear relaxation. Specifically, we prove that the existence of a multidimensional proportional apportionment is fully characterized by the fact that the linear relaxation of this integer linear program admits an integer optimal solution. We can then use this technique to establish that in general multidimensional proportional apportionments may fail to exist. This result is established by extending the network flow approach (6, 9) and conducting a careful primal–dual analysis. Furthermore, we use this approach to show that determining the existence of a proportional apportionment in the multidimensional setting (dimension 3 and higher) is a computationally hard problem (NP-complete). This is in sharp contrast with the two-dimensional case, in which this decision problem is polynomially solvable (2, 3). These results can be found in *Section 2*, and *SI Appendix*, section 1 contains their proofs.

Given that multidimensional proportional apportionments may fail to exist (and are in general hard to compute), we study what happens when we allow small violations in the marginals. Specifically, we consider the question of whether we can obtain an apportionment satisfying the proportionality condition by allowing it to violate the marginals by a certain amount and whether this can be done efficiently (polynomial in the house size). Our main result provides a positive answer to this question. Specifically, we prove that, if the nonnegative integers $u_1, \ldots, u_d$ are the target maximum violations in each dimension, a d-dimensional proportional apportionment exists so long as $\sum_{\ell=1}^{d} 1/(u_\ell + 2) \leq 1$.[‡,§] In dimension 2, with $u_1 = u_2 = 0$ this recovers the existence result of Balinski and Demange (2), while in dimension 3 a violation of one seat in each dimension is enough to guarantee existence. The main technical ingredient is to follow the lens of discrepancy theory, mainly the celebrated Beck–Fiala theorem (10). Furthermore, we provide a finer analysis on the deviations allowed by our algorithm, going beyond the typical ℓ_∞ analysis of deviations in discrepancy rounding of linear programs. To this end, we design an algorithm for an appropriate discrepancy problem in hypergraphs that might be of independent interest. Starting from an optimal solution of our base linear relaxation for the multidimensional problem, we run our discrepancy algorithm to get an apportionment with the desired deviations, and by using the structure of our linear program we show that proportionality is satisfied. These results can be found in *Section 3*, and *SI Appendix*, section 2 contains their proofs.

Finally, in *Section 4*, we test our method for finding a three-dimensional proportional apportionment in the context of the 2021 Chilean Constitutional Convention election. This election sought to elect a convention achieving proportionality across three dimensions: political, geographical, and gender. We observe that our method leads to an apportionment fulfilling the prescribed marginals and achieving exact gender parity. We also conclude that our method is significantly more representative than the one used. Finally, by simulating small random perturbations to the votes, we conclude that our approach is more robust in that these perturbations translate into only small changes in the house configuration.

Literature Overview. There is a rich literature body for the apportionment problem and the divisor methods, intersecting different areas such as operations research, computer science, and political science. For a formal treatment of the theory and a historical survey, we refer to the book of Balinski and Young (11) and to the recent book by Pukelsheim (12). For a deeper treatment of social choice and new methods, we also refer to the book and article by Balinski and Laraki (13, 14).

Biproportionality and matrix scaling. After Balinski and Demange (2, 3) first developed the biproportional method, some variants of it were later proposed by Balinski (15) and Balinski and González (16). Rote and Zachariasen (8) and Gaffke and Pukelsheim (6, 7) provided a unified view of biproportionality using network flow formulations. Pukelsheim et al. (17) also provided a wider view of network flow methods and their usage for electoral systems. The matrix-scaling problem has been studied extensively in the optimization, statistics, algorithms, and machine-learning communities and we refer to the survey by Idel (18) for an extensive treatment of this problem. Particularly relevant is the work by Sinkhorn (19) and subsequent complexity and algorithmic results by Sinkhorn and Knopp (20), Rothblum and Schneider (9), and Nemirovski and Rothblum (21). Kalantari et al. (22) analyzed an algorithm for matrix scaling introduced by Balinski and Demange (3), and very recently there have been several works on developing faster algorithms for matrix scaling and improved analysis of existing methods (23–25).

[†]These include Zurich, Aargau, Schaffhausen, Nidwalden, Zug, Schwyz, and Valais.

[‡]That is, in each dimension $\ell \in \{1, \ldots, d\}$ we allow the marginals to be additively violated by at most u_ℓ.

[§]The result actually requires a mild additional assumption that, for instance, is satisfied if the original vote matrix does not contain zeros.

Discrepancy theory. In the classic discrepancy minimization problem, there is a fractional vector x with entries in $[0,1]$ satisfying $Ax=b$ for some binary matrix A and an integer vector b, and the goal is to round x to get an integral vector $\tilde{x}$ with entries in $\{0,1\}$ in a way such that the maximum deviation $\|Ax - A\tilde{x}\|_\infty$ is as small as possible. In a celebrated result, Beck and Fiala (10) provided an algorithm to perform such rounding while achieving a maximum deviation of at most the maximum ℓ_1 norm of a column in A. This result was later improved for certain regimes (26–28). Also remarkable are the recent works by Bansal et al. (29–31), Lovett and Meka (32), and Rothvoss (28) that provide several algorithmic results for different discrepancy problems involving ℓ_∞ and ℓ_2 violations.

1. Preliminaries

In the classic apportionment problem, the input is given by a pair $(\mathcal{P}, H)$, where $\mathcal{P}$ is a vector with integer nonnegative entries containing the votes obtained by each party $i \in \{1,\dots,n\}$ and H is the house size, i.e., the total number of seats to allocate. The goal is to decide how many of the H seats should be given to each party. A feasible solution to this problem is formally described by a vector x such that $\sum_{i=1}^n x_i = H$ and x_i is a nonnegative integer for every $i \in \{1,\dots,n\}$, representing the number of seats allocated to party i. Clearly, a feasible solution always exists, but the challenge is to allocate the H seats proportionally to the votes. Naturally, seats cannot be divided fractionally, and therefore proportionality in this context needs to be defined appropriately. This paradigm is captured by a family of broadly used methods called divisor methods, which we formally describe in what follows.

A. Signpost Sequences, Rounding Rules, and Divisor Methods. Following the formalization introduced by Balinski and Young (11), a signpost sequence is a function defined over the nonnegative integers, $s : \mathbb{N} \to \mathbb{R}_+$, satisfying $s(0)=0$, $s(q) \in [q-1, q]$ for every strictly positive integer q, and the following left–right disjunction property: 1) If $s(p) = p-1$ for some $p \geq 2$, then $s(q) < q$ for every $q \geq 1$, and 2) if $s(q) = q$ for some $q \geq 1$, then $s(p) > p-1$ for every $p \geq 2$. In particular, any signpost sequence is strictly increasing over the strictly positive integers. To every signpost sequence s we can associate a rounding rule $[\![\cdot]\!]_s$ as follows: $[\![0]\!]_s = \{0\}$, $[\![t]\!]_s = \{q\}$ when $t \in (s(q), s(q+1))$ and $[\![t]\!]_s = \{q-1, q\}$ when $t = s(q) > 0$. Especially relevant are the signpost sequences of the form $s(q) = q - \Delta$ for every strictly positive integer q and some fixed $\Delta \in [0,1]$, as they capture the usual rounding operations. These signpost sequences are called stationary. To mention a few, $\Delta = 0$ corresponds to the classic downward rounding when t is fractional, since $s_1(q) = q$ for every $q \in \mathbb{N}$ implies, for example, that both 4.3 and 4.8 belong to the interval $(s_1(4), s_1(5)) = (4,\ 5)$ and thus $[\![4.3]\!]_{s_1} = [\![4.8]\!]_{s_1} = 4$. Similarly, $\Delta = 1/2$ coincides with the standard rounding when $t - 1/2$ is fractional, since defining $s_2(q) = q - 1/2$ for every $q \in \mathbb{N}$ we have, following the same example, that $4.3 \in (s_2(4), s_2(5)) = (3.5,\ 4.5)$ and $4.8 \in (s(5), s(6)) = (4.5,\ 5.5)$, and therefore $[\![4.3]\!]_{s_2} = 4$ and $[\![4.8]\!]_{s_2} = 5$.

The divisor method associated to a signpost sequence s works as follows: Given a pair $(\mathcal{P}, H)$, compute a vector $x \in \mathbb{N}^n$ with $\sum_{i=1}^n x_i = H$ for which there is a strictly positive value λ, called a multiplier, such that $x_i \in [\![\lambda \mathcal{P}_i]\!]_s$ for each $i \in \{1,\dots,n\}$. For every signpost sequence and every pair $(\mathcal{P}, H)$, the divisor method is guaranteed to provide a solution (11). In the context of voting, the classic Jefferson/D'Hondt method corresponds to the divisor method associated to the stationary signpost sequence with $\Delta = 0$. Other classic methods are the one by Webster/Sainte-Laguë, corresponding to the divisor method associated to the stationary signpost sequence with $\Delta = 1/2$, and the method by Adams, corresponding to the divisor method associated to the stationary signpost sequence with $\Delta = 1$. As a simple example, consider an instance with 10 seats to allocate and three political parties obtaining votes 129, 102, and 69. As depicted below, under the Jefferson/D'Hondt method the parties obtain 5, 3, and 2 seats, respectively, whereas the Webster/Sainte-Laguë method leads to an apportionment of 4, 4, and 2:

$$\mathcal{P} = \begin{pmatrix}129\\102\\69\end{pmatrix} \xrightarrow{\lambda=0.039} \lambda\mathcal{P} = \begin{pmatrix}5.03\\3.98\\2.69\end{pmatrix} \xrightarrow{[\![\cdot]\!]_{s_1}} x = \begin{pmatrix}5\\3\\2\end{pmatrix},$$

$$\mathcal{P} = \begin{pmatrix}129\\102\\69\end{pmatrix} \xrightarrow{\lambda=0.0347} \lambda\mathcal{P} = \begin{pmatrix}4.48\\3.54\\2.39\end{pmatrix} \xrightarrow{[\![\cdot]\!]_{s_2}} x = \begin{pmatrix}4\\4\\2\end{pmatrix}.$$

Observe that a key property of divisor methods is that a vote for one candidate favors the whole party, thus prioritizing voting for common ideas over personal candidates. This property still holds in the extension of divisor methods to multiple dimensions that follows. For an extensive treatment of the theory of divisor methods and their historical aspects, we refer to the books by Balinski and Young (11) and Pukelsheim (12).

B. Multidimensional Apportionment. Balinski and Demange (2, 3) extended the classic notion of proportionality captured by the divisor methods to the case in which the election is ruled by two dimensions, introducing the so-called biproportional method. These dimensions may represent, for example, the set of parties and the set of districts, and both the voting results and the apportionment can be written as a matrix where each row corresponds to a party and each column corresponds to a district. Intuitively, the idea in this case is to find one multiplier by row and one multiplier by column, such that when scaling the vote matrix according to these multipliers and rounding the result, we obtain an apportionment matrix where the sum of each row is equal to the number of seats assigned to the corresponding party and similarly, each column sums up to the amount of seats the corresponding district should get. In the previous example, let us suppose that the votes are cast in two different districts and that five seats should be allocated to the candidates running in each of them. The procedure for finding a biproportional apportionment for a specific vote matrix according to the downward rounding rule is illustrated below. As in the one-dimensional example, the first step corresponds to the scaling process, in this case multiplying each entry of the vote matrix by the multiplier associated to its row (on the right of the matrix) and by the multiplier associated to its column (below the matrix), and the second step corresponds to the rounding procedure:

$$\mathcal{V} = \begin{matrix}\begin{pmatrix}84 & 45\\69 & 33\\42 & 27\end{pmatrix} & \begin{matrix}0.07\\0.06\\0.07\end{matrix}\\ \begin{matrix}0.5 & 1\end{matrix} & \end{matrix} \longrightarrow \begin{pmatrix}2.94 & 3.15\\2.07 & 1.98\\1.47 & 1.89\end{pmatrix} \xrightarrow{[\![\cdot]\!]_{s_1}} x = \begin{pmatrix}2 & 3\\2 & 1\\1 & 1\end{pmatrix}.$$

It is important to note that the apportionment matrix specifies the number of seats assigned to each party in each district, so in this example the first party obtains two seats in the first district and three seats in the second district and similarly for each other party. Observe that the sums by row are exactly the one-dimensional apportionment $(5,3,2)$ and that both columns sum up to 5, as desired.

We now generalize this approach to arbitrary dimension in the natural way. In the d-dimensional apportionment problem, the input is given by a tuple $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ described as follows: For each $\ell \in \{1, \ldots, d\}$ we have a set N_ℓ such that $\mathcal{N} = \{N_1, \ldots, N_d\}$ and $\mathcal{V}$ is a vector with nonnegative integer entries in the Cartesian product $\prod_{\ell=1}^{d} N_\ell$. For each $\ell \in \{1, \ldots, d\}$ and each $v \in N_\ell$ there are integer values m_v^- and $m_v^+ > 0$ called lower and upper marginals, respectively, and H is a strictly positive integer value such that $\sum_{v \in N_\ell} m_v^- \leq H \leq \sum_{v \in N_\ell} m_v^+$ for every $\ell \in \{1, \ldots, d\}$. In the context of a political election, the values $\mathcal{V}_e$ represent the number of votes obtained by the tuple e, H represents the house size, and the value m^- (respectively m^+) defines a lower (upper) bound for the amount of seats that each element (party, district, etc.) should get.[¶] For an instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ we denote by $E(\mathcal{V})$ the subset of tuples $e \in \prod_{\ell=1}^{d} N_\ell$ such that $\mathcal{V}_e > 0$.

Given a d-dimensional instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ and a signpost sequence s, we say that $x \in \mathbb{N}^{E(\mathcal{V})}$ is a d-dimensional proportional apportionment if there exists a strictly positive value μ, and for every $\ell \in \{1, \ldots, d\}$ and every $v \in N_\ell$ there exists a strictly positive value λ_v, called a multiplier, such that the following holds:

$$m_v^- \leq \sum_{e \in E(\mathcal{V}): e_\ell = v} x_e \leq m_v^+ \quad \text{for every } \ell \in \{1, \ldots, d\} \text{ and every } v \in N_\ell, \qquad [1]$$

$$\sum_{e \in E(\mathcal{V})} x_e = H, \qquad [2]$$

$$s(x_e) \leq \mathcal{V}_e \cdot \mu \cdot \prod_{\ell=1}^{d} \lambda_{e_\ell} \leq s(x_e + 1) \quad \text{for every } e \in E(\mathcal{V}), \qquad [3]$$

and furthermore, we have the following conditions regarding the values of the multipliers for every $\ell \in \{1, \ldots, d\}$ and every $v \in N_\ell$,

$$\text{If } \lambda_v > 1, \text{ then we have } \sum_{e \in E(\mathcal{V}): e_\ell = v} x_e = m_v^-, \qquad [4]$$

$$\text{If } \lambda_v < 1, \text{ then we have } \sum_{e \in E(\mathcal{V}): e_\ell = v} x_e = m_v^+. \qquad [5]$$

We denote by $\mathcal{A}_s(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ the set of triplets (x, μ, λ) where x is integral, μ is strictly positive, λ is strictly positive in each entry, and the triplet satisfies conditions **1–5**. Note that when $s(1) = 0$, any x satisfying condition **3** must be strictly positive in each entry. In addition, given the strict positivity of the values μ and λ_v, condition **3** is equivalent to $x_e \in [\![\mathcal{V}_e \cdot \mu \cdot \prod_{\ell=1}^{d} \lambda_{e_\ell}]\!]_s$; thus, it captures the idea of proportionality. We remark that for $d = 2$ this corresponds to the proportionality notion of Balinski and Demange, in the sense that their definition of a biproportional apportionment is equivalent to our definition of a two-dimensional proportional apportionment.

In this type of method, the marginals m_v^-, m_v^+ have to be determined for each element v, and the votes $\mathcal{V}_e$ must be known for each tuple e. In the case of the marginals, they might come either from values previously defined by law, as in the case of districts or of gender parity, or from the vote itself through solving a one-dimensional apportionment, as in the case of political parties or lists. On the other hand, the ballot has to be designed in a way that the entries of the tensor $\mathcal{V}$ can be computed. For example, the Chilean Constitutional Convention election was a single-vote election where on each district the ballot contained the full set of candidates of every political party or list. Then, from the voting data of this election we can construct a three-dimensional instance where the dimensions are given by the districts, gender, and political lists. In general, the information may come both from demographic attributes of the voter, as in the case of the districts or as would be natural in the case of considering ethnicity as a dimension, and from the declared preferences, as could be the case for the list or gender. The latter requires that the ballot is informative enough on these dimensions, which is straightforward in open-list systems where people vote for a single candidate, but may require adaptations in others, such as dividing possible closed lists among different genders.

[¶]Note that, in the two-dimensional example above, we consider single values and not lower and upper bounds for the marginals. This is captured in this general model by simply setting $m^- = m^+$.

2. A Linear Programming Approach

In this section, we introduce an integer linear program inspired by transportation and matrix-scaling problems. We prove a correspondence between the integer optimal solutions of its linear relaxation and the multidimensional proportional apportionments. Using this characterization, we show the inexistence of proportional apportionments for some instances of the problem and the computational hardness of deciding the existence of such apportionments.

A. An Integer Linear Program Inspired by Matrix Scaling. We follow a related network flow approach introduced by Rote and Zachariasen (8) for matrix scaling and used by Gaffke and Pukelsheim (6, 7) to model biproportional apportionments. Our integer linear program to study the d-dimensional apportionment problem is constructed as follows: Consider a d-dimensional instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ and a signpost sequence s. For each $e \in E(\mathcal{V})$ and each $t \in \{1, \ldots, H\}$ we have a binary variable y_e^t and its cost in the objective function is given by $\log(s(t)/\mathcal{V}_e)$ if $s(t) > 0$ and zero otherwise:

$$\min \sum_{e \in E(\mathcal{V})} \sum_{\substack{t \in \{1,\ldots,H\}: \\ s(t) > 0}} y_e^t \log\left(\frac{s(t)}{\mathcal{V}_e}\right) \qquad [6]$$

$$\text{s.t.} \quad \sum_{t=1}^{H} y_e^t = x_e \quad \text{for every } e \in E(\mathcal{V}), \qquad [7]$$

$$\sum_{e \in E(\mathcal{V})} x_e = H, \qquad [8]$$

$$\sum_{e \in E(\mathcal{V}): e_\ell = v} x_e \geq m_v^- \quad \text{for every } \ell \in \{1, \ldots, d\} \text{ and every } v \in N_\ell, \qquad [9]$$

$$\sum_{e \in E(\mathcal{V}): e_\ell = v} x_e \leq m_v^+ \quad \text{for every } \ell \in \{1, \ldots, d\} \text{ and every } v \in N_\ell, \qquad [10]$$

$$y_e^1 \geq \lfloor 1 - s(1) \rfloor \quad \text{for every } e \in E(\mathcal{V}), \qquad [11]$$

$$y_e^t \in \{0, 1\} \quad \text{for every } e \in E(\mathcal{V}) \text{ and every } t \in \{1, \ldots, H\}. \qquad [12]$$

The variable x_e represents the total number of seats to be allocated in the apportionment for the tuple e and constraint **7** takes care of aggregating the seats in these variables. Constraint **8** ensures to respect the house size and constraints **9** and **10** enforce every feasible solution to satisfy the marginals. Finally, constraint **11** ensures $x_e \geq 1$ if $s(1) = 0$. We remark that this integer linear program can be equivalently written by omitting the variables y at the price of having a nonlinear convex objective.

B. Characterizing Optimal Solutions of the Linear Relaxation. When $d = 2$, the above problem is as hard as a transportation problem in a bipartite network, and in consequence, one can recover an optimal solution of this problem by solving the linear relaxation. Furthermore, it can be shown when $d = 2$ that any optimal extreme point defines a proportional apportionment, where multipliers are obtained by computing the exponential of the corresponding dual solution (6, 8, 12). Therefore, in the general d-dimensional setting, the first question that we address is the following: Can we characterize the set of proportional apportionments in terms of the set of optimal solutions of the linear relaxation of [**6**]–[**12**]? The following result provides a positive answer to this question:

Theorem 1. *Let* $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ *be an instance of the d-dimensional apportionment problem, let s be a signpost sequence, and let* $x \in \mathbb{N}^{E(\mathcal{V})}$. *Then, there exist* μ *and* λ *such that* $(x, \mu, \lambda) \in \mathcal{A}_s(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ *if and only if there exists y such that* (x, y) *is an optimal solution for the linear relaxation of* [**6**]–[**12**].

This result is established by studying the optimality conditions of the linear relaxation of [**6**]–[**12**] and has as an immediate implication that there exists a d-dimensional proportional apportionment if and only if the linear relaxation of [**6**]–[**12**] admits an integer optimal solution. This provides a natural way of studying the existence of such apportionments for a given instance: solving the program [**6**]–[**12**] and its linear relaxation, and comparing the objective values. If they coincide, any optimal solution of the integer linear program defines a proportional apportionment; otherwise, there is no such apportionment. We use this procedure in the following subsection to show that there are instances that do not admit proportional apportionments, and in *Section 2.d* we prove that, unless $\mathsf{P} = \mathsf{NP}$, there is no efficient algorithm to decide whether this is the case for a given instance.

C. Nonexistence of Three-Dimensional Proportional Apportionments. We recall that the existence of d-dimensional proportional apportionments when $d \in \{1, 2\}$ is completely understood, and necessary and sufficient conditions are provided in general (2, 3). In particular, when the apportionment instance is two-dimensional and $\mathcal{V}$ is strictly positive, there is always a proportional apportionment. This follows from the fact that when $d = 2$, the linear relaxation of [**6**]–[**12**] is integral, as a consequence of total unimodularity, and the feasibility of this program is guaranteed by $\mathcal{V}$ being strictly positive. For practical purposes, this is relevant since it guarantees the existence of proportional apportionments for a fairly natural setting. Then, the following question arises naturally: Given a d-dimensional instance with $d \geq 3$ and $\mathcal{V}$ strictly positive, can we always find a proportional apportionment? We answer this question in the negative when s belongs to the relevant family of stationary signpost sequences.

Theorem 2. *There exists an instance* $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ *of the three-dimensional apportionment problem, with* $\mathcal{V}$ *strictly positive in each of its entries, such that for every stationary signpost sequence s we have* $\mathcal{A}_s(\mathcal{N}, \mathcal{V}, m^-, m^+, H) = \emptyset$.

This result is proved in a three-dimensional instance where $|N_\ell| = 2$ for each $\ell \in \{1, 2, 3\}$, and $\mathcal{V}_e > 0$ for every $e \in N_1 \times N_2 \times N_3$. This shows that even for very small instances with $\mathcal{V}$ strictly positive, the existence is not guaranteed.

D. Complexity of the Multidimensional Apportionment Problem. So far, we know that a proportional apportionment for a given instance is always optimal for the linear relaxation of [**6**]–[**12**], and moreover, it defines an extreme point of its feasible region. The natural question is how to determine the existence of an integer extreme point of this optimal region or to ensure that no such point exists. More formally, consider the following decision problem: Given an instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ of the d-dimensional apportionment problem and a signpost sequence s, decide whether $\mathcal{A}_s(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ is empty or not. Within this subsection, we refer to this decision problem as the (d, s) -proportional apportionment problem. We remark that the work of Balinski and Demange shows that the $(2, s)$-proportional apportionment problem can be solved in polynomial time for every signpost sequence s (2, 3). Then, the natural question that arises is the following: What is the complexity of the (d, s)-proportional apportionment problem when $d \geq 3$? The following is our main result in this line:

Theorem 3. *For every signpost sequence s and every* $d \geq 3$, *the* (d, s) *-proportional apportionment problem is* NP*-complete.*

We prove this theorem by a hardness reduction from the perfect matching problem in d-partite hypergraphs. Recall that $G = (P, F)$ is a d-partite hypergraph if the set of vertices P can be partitioned into d disjoint sets $P_1, \ldots, P_d$, and every hyperedge $f \in F$ intersects each of the parts exactly once; that is, $|f \cap P_\ell| = 1$ for every $f \in F$ and every $\ell \in \{1, \ldots, d\}$. A 2-partite hypergraph is just a bipartite graph. We say that $F' \subseteq F$ is a perfect matching of G if for every $v \in P$ we have $|\{f \in F' : v \in f\}| = 1$. The problem of determining whether a d-partite hypergraph contains a perfect matching is NP-complete even when $d = 3$ and $|P_1| = |P_2| = |P_3|$, which is known as the three-dimensional matching problem (33).

As a remark, if one considers the case where $|N_\ell|$ is constant for every $\ell \in \{1, \ldots, d\}$, there exists a polynomial algorithm for the (d, s)-proportional apportionment problem: One can enumerate every possible base defining an extreme point of the linear relaxation of [**6**]–[**12**], and therefore we can check if there exists an optimal integer extreme point. *Theorem 1* guarantees the correctness of this algorithm.

3. A Linear Program Rounding Algorithm for Multidimensional Apportionment

In the previous section, we have addressed the multidimensional apportionment problem from an existence and complexity point of view. In particular, we have seen that there exist d-dimensional instances for which it is not possible to simultaneously satisfy conditions **1**–**5**. Therefore, in this section we address the following question: Is it possible to compute a vector that satisfies condition **3** and such that the violation in the other conditions is under control? We provide a positive answer to this question, summarized in the following theorem:

Theorem 4. *Let* $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ *be an instance of the d-dimensional apportionment problem and let s be a signpost sequence such that the linear relaxation of* [**6**]–[**12**] *is feasible. Let* $u_1, \ldots, u_d$ *be nonnegative integer values such that* $\sum_{\ell=1}^{d} 1/(u_\ell + 2) \leq 1$. *Then, there exists an integral vector* $X \in \mathbb{N}^{E(\mathcal{V})}$ *such that the following holds*:

1) $m_v^- - u_\ell \leq \sum_{e \in E(\mathcal{V}): e_\ell = v} X_e \leq m_v^+ + u_\ell$ *for every* $\ell \in \{1, \ldots, d\}$ *and every* $v \in N_\ell$.
2) *There exist* $\mu > 0$ *and a vector* λ *with strictly positive entries such that*

i) $s(X_e) \leq \mathcal{V}_e \cdot \mu \cdot \prod_{\ell=1}^{d} \lambda_{e_\ell} \leq s(X_e + 1)$ *for every* $e \in E(\mathcal{V})$.

ii) *For every* $\ell \in \{1, \ldots, d\}$ *and every* $v \in N_\ell$, *if* $\lambda_v > 1$, *then* $|\sum_{e \in E(\mathcal{V}): e_\ell = v} X_e - m_v^-| \leq u_\ell$.

iii) *For every* $\ell \in \{1, \ldots, d\}$ *and every* $v \in N_\ell$, *if* $\lambda_v < 1$, *then* $|\sum_{e \in E(\mathcal{V}): e_\ell = v} X_e - m_v^+| \leq u_\ell$.

Furthermore, X *can be found in time polynomial in* $|E(\mathcal{V})|$, $\sum_{\ell=1}^{d} |N_\ell|$, *and H.*

We have seen that there exist instances for the d-dimensional apportionment problem with $d \geq 3$ for which there are no integral optimal solutions for the linear relaxation of [**6**]–[**12**]. In contrast, we show that it is possible to round an optimal fractional solution in a way that the obtained integral vector satisfies the proportionality condition **3** and at the same time the violation in the marginals condition **1**, as well as in the multipliers conditions **4** and **5**, is under control. To do so, we study in *Section 3.A* a particular discrepancy problem in hypergraphs, inspired by the work of Beck and Fiala (10) for the discrepancy minimization problem. We present the algorithm necessary for *Theorem 4* and a brief analysis of this result in *Section 3.B*. We remark that the feasibility of the linear relaxation of [**6**]–[**12**] is ensured under mild assumptions, for instance, that each entry of $\mathcal{V}$ is strictly positive.

Note that when $d = 2$, *Theorem 4* is valid for $u_1 = u_2 = 0$; therefore it implies the existence of two-dimensional proportional apportionments whenever the linear relaxation of [**6**]–[**12**] is feasible, i.e., whenever there exists a vector in $\mathbb{R}_+^{E(\mathcal{V})}$ respecting the marginals and preserving the zeros of $\mathcal{V}$, with the additional condition that $\mathcal{V}_e > 0$ implies a strictly positive entry if $s(1) = 0$. This is one of the main results of Balinski and Demange (2), so *Theorem 4* can be seen as a generalization of their existence result for the case of arbitrary dimension.

A. A Discrepancy Problem in *d*-Partite Hypergraphs. Consider a d-partite hypergraph G with vertex partition $\{P_1, \ldots, P_d\}$ and hyperedges E, and let $x \in [0,1]^E$ be such that $\sum_{e \in \delta(v)} x_e$ is integral for every vertex v of G, where $\delta(v)$ is the set of hyperedges containing v. We are also given d nonnegative integer values $u_1, \ldots, u_d$ and the goal is to round x into an integral vector in $\{0,1\}^E$ in a way such that the deviation from $\sum_{e \in \delta(v)} x_e$ on every vertex $v \in P_\ell$ is at most u_ℓ for each $\ell \in \{1, \ldots, d\}$. Naturally, when $u_1 = \cdots = u_d$, we fall into the classic discrepancy minimization approach. In contrast, we are interested in providing a fine upper bound on the deviation throughout the different parts of the hypergraph. The following is our main result in this line:

Theorem 5. *Let G be a d-partite hypergraph with vertex partition* $\{P_1, \ldots, P_d\}$ *and hyperedges E. Let* $x \in [0,1]^E$ *be such that* $\sum_{e \in \delta(v)} x_e$ *is integral for every vertex v of G and let* $u_1, \ldots, u_d$ *be nonnegative integers such that* $\sum_{\ell=1}^{d} 1/(u_\ell + 2) \leq 1$. *Then, there exists* $z \in \{0,1\}^E$ *such that for every* $\ell \in \{1, \ldots, d\}$ *and every* $v \in P_\ell$ *it holds* $|\sum_{e \in \delta(v)} (z_e - x_e)| \leq u_\ell$, *and* $z_e = x_e$ *when* x_e *is integer. Furthermore,* z *can be computed in time polynomial in* $|E|$ *and* $\sum_{\ell=1}^{d} |P_\ell|$.

Algorithm 1. Iterative rounding algorithm:

Require: A d-partite hypergraph G with vertex partition $\{P_1, \ldots, P_d\}$ and hyperedges E, a vector $x \in [0,1]^E$, and nonnegative integer values $u_1, \ldots, u_d$.

Ensure: Binary vector $z \in \{0,1\}^E$.

1) Initialize $y^0 \leftarrow x$ and let $\mathcal{E}^0 = \{e \in E : y_e^0 \text{is fractional}\}$.
2) For each $\ell \in \{1, \ldots, d\}$, let $Q_\ell^0 = \{v \in P_\ell : |\delta(v) \cap \mathcal{E}^0| \geq u_\ell + 2\}$.
3) Let $z_e = y_e^0$ for every $e \notin \mathcal{E}^0$ and initialize $t \leftarrow 0$.
4) **While** there exists $\ell \in \{1, \ldots, d\}$ such that $Q_\ell^t \neq \emptyset$ **do**
5) Compute an extreme point y^{t+1} of $\mathcal{K}(y^t, \mathcal{E}^t, Q^t)$.
6) Let $\mathcal{E}^{t+1} = \{e \in E : y_e^{t+1} \text{is fractional}\}$.
7) For each $\ell \in \{1, \ldots, d\}$, let $Q_\ell^{t+1} = \{v \in P_\ell : |\delta(v) \cap \mathcal{E}^{t+1}| \geq u_\ell + 2\}$.
8) Let $z_e = y_e^{t+1}$ for every $e \in \mathcal{E}^t \setminus \mathcal{E}^{t+1}$. Update $t \leftarrow t + 1$.
9) Let T be the value of *t* that did not satisfy the loop condition.
10) Let $z_e \in \{\lfloor y_e^T \rfloor, \lceil y_e^T \rceil\}$ for every $e \in \mathcal{E}^T$.
11) Return *z*.

We present an iterative rounding algorithm, inspired by the classic discrepancy minimization result by Beck and Fiala (10) and formally described in *Algorithm 1*, that computes a solution *z* satisfying the conditions guaranteed by *Theorem 5*. To present this procedure, we introduce a simple linear program that is used during its execution. Given a vector $Y \in [0,1]^F$ with $F \subseteq E$, a subset of edges $\mathcal{E} \subseteq F$, and a subset of vertices $Q_\ell \subseteq P_\ell$ for each $\ell \in \{1, \ldots, d\}$, we consider the following linear program with variables y_e for each $e \in \mathcal{E}$:

$$\sum_{e \in \delta(v) \cap \mathcal{E}} y_e = \sum_{e \in \delta(v) \cap \mathcal{E}} Y_e \quad \text{for every } v \in \bigcup_{\ell=1}^{d} Q_\ell, \qquad [\mathbf{13}]$$

$$0 \leq y_e \leq 1 \quad \text{for every } e \in \mathcal{E}. \qquad [\mathbf{14}]$$

We denote by $\mathcal{K}(Y, \mathcal{E}, Q)$ the polytope of feasible solutions for this linear program.

Algorithm 1 iteratively solves a linear program in the form of [**13**] and [**14**]. The condition in the loop guarantees that the algorithm makes progress in fixing at least one new variable into a binary value. Once the loop condition is not satisfied, the algorithm rounds up or down the rest of the fractional variables and its output satisfies the properties guaranteed by *Theorem 5*.

It is worth mentioning two observations. First, note that the last step of *Algorithm 1* allows to round the remaining fractional entries as desired. Although the most natural way to minimize deviations might be to fix each entry to the nearest integer, this does not allow an improvement of the worst-case bound. The second observation is that the bound can be slightly refined for particular cases, when at least one of the parts in the vertex partition is small. In particular, the sufficient condition over integers $u_1, \ldots, u_d$ in *Theorem 5* can be replaced by $\sum_{\ell=1}^{d} \min\{\lfloor q/(u_\ell + 2) \rfloor, |P_\ell|\} < q$ for every strictly positive integer q. We remark that this is not a stronger version of *Theorem 5*, since the inequality over integers $u_1, \ldots, u_d$ becomes strict in the new condition.

B. Rounding an Optimal Solution of the Linear Relaxation. To obtain the result stated in *Theorem 4*, we present our *Algorithm 2*, which is based on two key steps. In the first step, we solve the linear relaxation of [**6**]–[**12**] to get a solution. If this solution is integral, then this is the output of the algorithm. Otherwise, we use this solution to feed our *Algorithm 1* and get an integral vector. Let α be such that $\alpha(e) = \{e_1, \ldots, e_d\}$ for each $e \in E(\mathcal{V})$. The function α captures the natural representation of the (ordered) tuples in $E(\mathcal{V})$ as (unordered) sets. We use this representation to go from the apportionment setting to the hypergraph representation.

Algorithm 2. Apportionment rounding algorithm:

Require: A d-dimensional instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ and $u_1, \ldots, u_d$ with $\sum_{\ell=1}^{d} 1/(u_\ell + 2) \leq 1$.

Ensure: An integral vector $X \in \mathbb{N}^{E(\mathcal{V})}$.

1) Let $(x^\star, y^\star)$ be an optimal solution of the linear relaxation of [**6**]–[**12**] in instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$.
2) If $x^\star$ is integral, **return** $X \leftarrow x^\star$.
3) If $x^\star$ is fractional, consider the d-partite hypergraph G with vertex partition $N_1, \ldots, N_d$ and hyperedges $\alpha(E(\mathcal{V}))$. Run *Algorithm 1* over the hypergraph G, the fractional vector w defined as $w_{\alpha(e)} = x_e^\star - \lfloor x_e^\star \rfloor$ for every $e \in E(\mathcal{V})$, and the values $u_1, \ldots, u_d$ and let $z \in \{0,1\}^{\alpha(E(\mathcal{V}))}$ be its output.
4) Return $X_e = \lfloor x_e^\star \rfloor + z_{\alpha(e)}$ for every $e \in E(\mathcal{V})$.

Theorem 4 allows constructing vectors that satisfy [**3**] with fixed maximum marginal deviations in some relevant cases, in particular, when d is fixed. For example, when $d = 3$, *Algorithm 2* can be run using any vector $\boldsymbol{u}$ in $\{(0,1,4), (0,2,2), (1,1,1)\}$. These vectors and their permutations actually constitute the Pareto frontier of deviations satisfying the sufficient condition in the theorem, which is particularly relevant in the case of elections, since it allows the incorporation of a new feature on top of the classic party and district dimension. We also remark that when $d \geq 3$, *Algorithm 2* can be run using $u = (d-2, \ldots, d-2)$. When one dimension has only two options (assume this is the first dimension), following the logic of the discussion after *Theorem 5* one can achieve even better bounds, like $u = (0,1,3)$ when $d = 3$ and $u = (0,2,3,4)$ when $d = 4$, instead of $u = (0,2,3,18)$, which is part of the Pareto frontier in the general case.

Finally, we remark that the bounds obtained cannot be strictly improved, in the sense that if we denote $f(u) = \sum_{\ell=1}^{d} 1/(u_\ell + 2)$, there is no function $g < f$ such that $g(u) \leq 1$ ensures the existence of a multidimensional proportional apportionment with deviation at most u_ℓ in each dimension $\ell \in \{1, \ldots, d\}$. In particular, we prove that in the case $d = 3$ it is not possible to ensure a maximum deviation given by $u = (0, 0, K)$ for any constant K, and we extend this impossibility to the case of higher dimension by induction. We leave as an open question whether there are other vectors $\boldsymbol{u}$ with $f(u) > 1$, for instance, $u = (0,1,1)$, defining deviations that are reachable for every instance of the problem, i.e., whether our sufficient condition for $\boldsymbol{u}$ defining feasible deviations is also necessary.

4. Results from the Chilean Constitutional Convention

In this last section, we test our method for the case of three dimensions, namely political lists, districts, and genders, using the downward rounding rule. List marginals are calculated according to the votes obtained by each list through a single-dimensional Jefferson/D'Hondt method, and district marginals are predefined by law, and gender marginals ensure parity, i.e., 50% of the seats to each gender. We refer to this as the three-proportional method (TPM) in the following. The testing ground is provided by the recent election of the Chilean Constitutional Convention (May 15 to 16, 2021), and the basis of the comparison is given by the Constitutional Convention method (CCM). Chile's electoral map is divided into 28 electoral districts with a specified number of seats to be allocated in each district. In total, 155 seats were to be allocated, 17 of which were reserved for ethnic minority groups, so that 138 seats were allocated to the 28 districts. Our comparison considers only these nonethnic seats.[#] We also mention that each voter votes for at most one candidate of the voter's district.

In the recent Chilean Constitutional Convention election a total of 70 lists, including over 1,300 candidates, competed for these 138 seats. Three of these lists corresponded to well-established political alliances. The XP list represented the right-wing parties, including not only the traditional parties *Renovación Nacional* and *Unión Demócrata Independiente*, but also the newer centrist *Evopoli* and the extreme right *Partido Republicano*. The YB list represented the center-left parties that have mostly governed Chile in the last three decades, including the *Democracia Cristiana* and the *Partido Socialista*. The third list is the YQ list and corresponds to the left-wing parties such as the *Partido Comunista* and a number of much newer parties. Additionally, there were two important politically independent players in the election that arose as conglomerates encompassing different lists (which did not compete in any district). These correspond to what we denote by LP (*Lista del Pueblo*) and INN (*Independientes No Neutrales*).[‖,**] By observing the outcome of both TPM and CCM, we have that among the 70 lists and 28 independent candidates, only 20 lists and one independent candidate obtain enough votes to be elected in either system.[††] For ease of exposition, when presenting the results, we omit the votes of the other lists and independent candidates, none of which obtained more than 0.51% of the votes and jointly represent less than 10% of them. Note that the results are not affected by this modification.

In what follows we compare the CCM results with what would have happened if TPM was in place.[‡‡] To this end let us first describe CCM, which works as follows: In the first step, the seats of each district are divided between the lists and independent candidates according to the single-dimensional Jefferson/D'Hondt method, using the votes obtained by all the candidates of each list. Then, the seats assigned to each list are divided between its sublists (usually political parties) through the same method and provisionally assigned to the candidates of these sublists with more individual votes. If at this point the set of elected candidates achieves gender balance, i.e., the same number of men and women if the number of seats of the district is even and at most one more man/woman if it is odd, the seats are assigned to these candidates. Otherwise, the following procedure is repeated until the gender balance condition is satisfied: Pick the provisionally elected candidate of the overrepresented gender with the lowest number of votes, and assign in the candidate's place the provisionally nonelected candidate of the other gender and the candidate's same sublist (or list, in case the former is not possible) with the highest number of individual votes. Observe that both CCM and TMP implicitly require that there are enough candidates of each gender.

Political Balance. As a quality measure of an apportionment, we consider the deviation of the political distribution from the perfectly fair distribution, a.k.a. fair share in the literature, which

[#] Ethnic seats were assigned through a parallel election without taking into account political or geographical considerations, so considering them as a fourth dimension, which would be a natural further application of multidimensional proportionality, would not be possible in terms of data availability and would not allow a comparison with the actual results.

[‖] This association is standard as reported, for instance, by https://2021.decidechile.cl/#/ev/2021. Full election data can be found on the website of the Chilean Servicio Electoral (SERVEL) https://www.servel.cl/.

[**] When presenting the results for the remaining lists, we use the election codes.

[††] This independent candidate is denoted as IND9 because of the number of the district where he participated.

[‡‡] Further comparisons of CCM with proportional methods were recently presented in ref. 34.

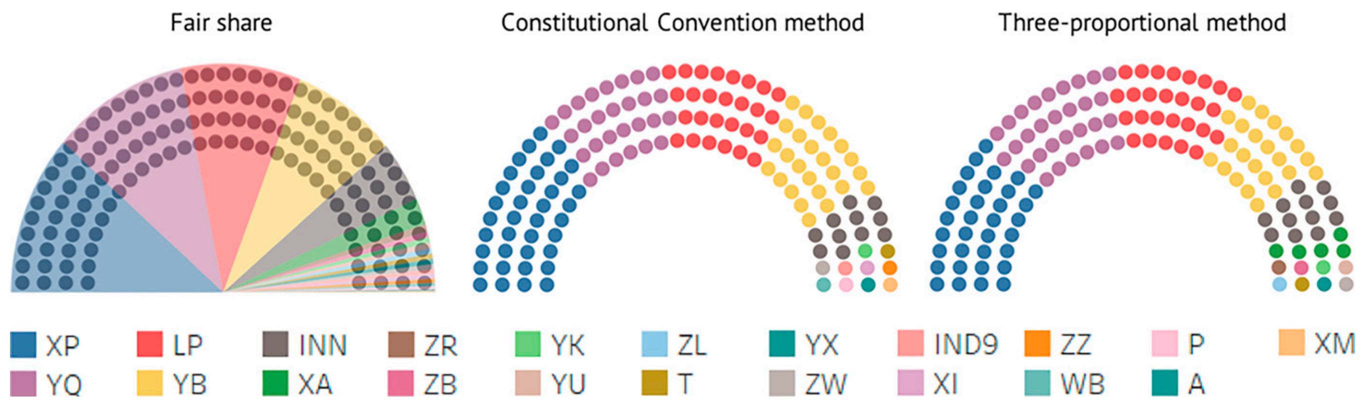

Fig. 1. Political distribution by method and fair share.

assigns to each list the (possibly fractional) number of seats that corresponds to the proportion of the house that the votes obtained represent. Fig. 1 shows the proportion obtained by each list with the 138 seats as a background, to give a graphical idea of the fair share, and the political distribution of the Constitutional Convention under both methods.[§§] It is observed that TPM generates a political distribution much closer to the fair share than the Constitutional Convention method, with a smaller overrepresentation of the most voted list and an assignment of seats to the 14 top-voted lists. It is particularly relevant to remark that CCM does not assign any seat to list XA, which is the sixth most voted list with almost 4% of the votes, while TPM allocates five seats to this list.

The notion of closeness or dispersion with respect to the fair share can be formalized through the Euclidean distance between an apportionment and the fair share or, in other words, the SD of an apportionment with respect to the fair share. This notion is easily extended to the apportionment of a single district as well, comparing the political distribution of the seats assigned in the district with the fair distribution according to the votes.[¶¶] Fig. 2 shows the SD of the apportionments obtained with each method by district. (The data used for Fig. 2 are in *SI Appendix*, Table S4.)

Naturally, CCM is locally closer to the fair share than TPM, which is essentially a property of the design since CCM achieves local proportionality. However, when summing up the results by district, the local errors generated by CCM start to add up and the distortion with respect to the fair share increases. On the other hand, TPM is designed to achieve global proportionality so that the national results are much closer to the fair share of the vote. In fact, the SD of the apportionment obtained with TPM with respect to the fair share is 2.49, and with CCM this value is 6.44. There is, therefore, a trade-off between local and global political representation.

Robustness. Another criterion we use to compare CCM and TPM is their robustness to small perturbations in the votes. To evaluate this aspect, we conduct $n = 1,000$ simulations, and in each one we multiply the votes obtained by each candidate by a normally distributed value with mean 1 and SD 0.05. We then compute the distribution of the number of seats transferred from one list to any other on each simulation starting from the original apportionment. Denoting the seats obtained by each list $\ell \in L$ in the original apportionment as y_ℓ, the seats obtained by each list $\ell \in L$ in simulation $i \in \{1, \ldots, n\}$ as y_ℓ^i, and the variable of interest as T^i, this variable is given by

$$T^i = \frac{1}{2} \sum_{\ell \in L} |y_\ell - y_\ell^i|.^{\#\#}$$

Fig. 3 plots the distribution of this variable under each method. Since the three-proportional method assigns to each list a number of seats determined by its total votes, instead of the votes

[§§]The data used for Fig. 1 are in *SI Appendix*, Table S1.

[¶¶]District data can be found in *SI Appendix*, Tables S2 and S3.

[##]Note that since seat transfers are counted twice in the summation, we divide the expression by 2.

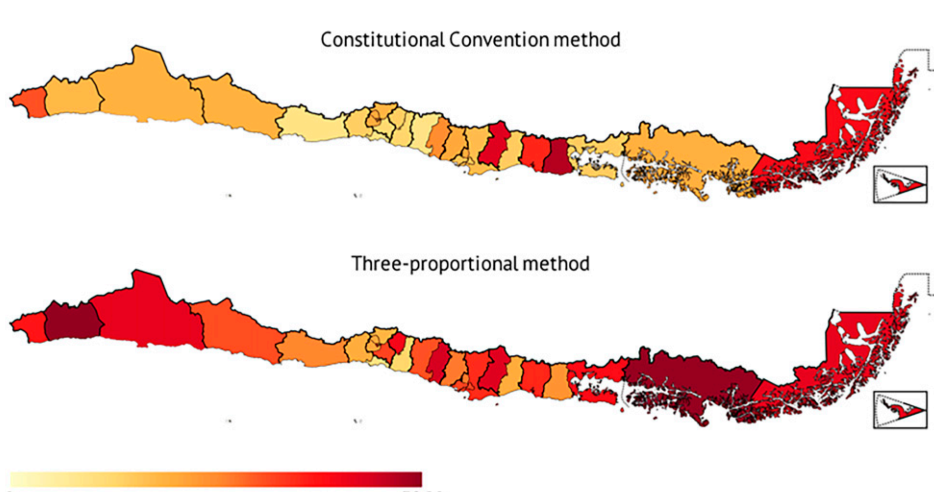

Fig. 2. SD of political distribution with respect to the fair share by method and district.

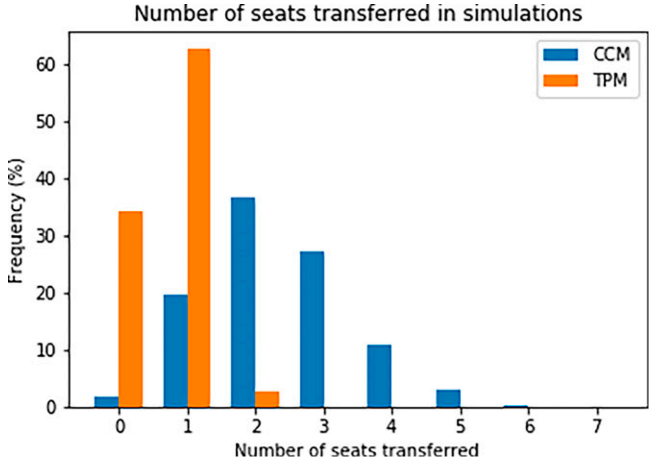

Fig. 3. Distribution of the seats transferred between lists under perturbations of the votes.

by district, this method generates a more robust result in the face of changes in the votes. Indeed, the average number of seats transferred under this method is 0.68, and this value increases to 2.37 under the Constitutional Convention method. When evaluating robustness locally, the number of seats transferred from one list to another, district by district, as a result of the random perturbations is (again naturally) smaller in the case of CCM, with an average of 3.28 seats against 5.36 in the case of TPM.

Gender Balance. CCM leads to a house composed of 70 men and 68 women, while an absolute gender balance of 69 men and 69 women is achieved by TPM. It is important to note that CCM guarantees the same number of elected men and women in districts with an even number of seats and a difference of at most one for those with an odd amount of seats, thus performing better in a local sense, with an average difference of 0.5 seats between genders against an average difference of 1.79 in the case of TPM. However, these deviations might add up and could lead, in the worst case, to a difference of 14 seats between men and women with the district configuration used for this election. On the other hand, global gender balance is guaranteed by the definition of TPM whenever deviations are not necessary, which is the case both for the actual election results and for all the simulations described in the previous paragraph. The trade-off between local and global representation arises again in the case of gender.

The Value of a Vote. As a final observation, we remark that when using TPM each vote is equally valuable in favor of the chosen list, because its number of seats, as was pointed out, depends only on its total number of votes. In the Constitutional Convention election, comparing the number of seats assigned to each district and the people who voted in each of them, the votes of some people were 5.44 times more valuable than the votes of people living in a different district (this ratio becomes greater than 1,000 when considering the seats reserved to ethnic groups). Incorporating geographic division, gender, and possibly other criteria as additional dimensions instead of making separate elections for each clearly allows getting closer to the well-known principle of one person, one vote.

In conclusion, our experiments show that TPM provides a well-balanced, near-to-proportional global representation in all relevant dimensions, in addition to ensuring robustness and equal value of each vote in terms of the global apportionment. The natural price of this is some imprecision at the local level when comparing with local methods. Therefore, applications of multidimensional proportionality may be of special interest for elections of representative bodies whose main impact is at the national level.

Data Availability. Previously published data were used for this work (https://www.servelelecciones.cl/).

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